Lecture 3

Recall:

Let X be a nonempty set

Def: A nonempty collection of sets $\mathcal{A} \subseteq 2^X$ is an algebra if it is closed under finite unions and complements.

Lemma: If $\mathcal{A} \subseteq 2^X$ is an algebra,
① $\emptyset \in \mathcal{A}$, $X \in \mathcal{A}$
② $\mathcal{A}$ is closed under finite intersections

Def: $\mathcal{A} \subseteq 2^X$ is a <u>σ-algebra of subsets of X</u> if it is an algebra and if it is closed under countable unions.

Def: Given a nonempty set X and a σ-algebra $\mathcal{M} \subseteq 2^X$, we will call $(X, \mathcal{M})$ a <u>measurable space</u>. We will call $E \in \mathcal{M}$ a <u>measurable set</u>.

Prop: Given any $\mathcal{E} \subseteq 2^X$, there exists a smallest σ-algebra containing $\mathcal{E}$, which we denote $\mathcal{M}(\mathcal{E})$ and refer to as the <u>σ-algebra generated by $\mathcal{E}$</u>.

Recall: a topology $\mathcal{T}$ on X is a collection of subsets of X that contains $\emptyset$ and X and is closed under arbitrary unions and finite intersections. Elements of $\mathcal{T}$ are called "open sets".

Let $(X, \mathcal{T})$ be a topological space.

Def: The Borel σ-algebra of X, denote $\mathcal{B}(X)$, is the σ-algebra generated by $\mathcal{T}$. Its members are known as Borel sets.

Prop: The Borel σ-algebra of $\mathbb{R}$, denote $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following

(i) open intervals, $\mathcal{E}_1 = \{(a,b) : a < b\}$

(ii) closed " ", $\mathcal{E}_2 = \{[a,b] : a \leq b\}$

(iii) half-open " ", $\mathcal{E}_3 = \{[a,b) : a < b\}$

(iv) open rays, $\mathcal{E}_4 := \{(a,+\infty) : a \in \mathbb{R}\}$

(v) closed rays, $\mathcal{E}_5 := \{[a,+\infty) : a \in \mathbb{R}\}$

Def: Given a measurable space $(X, \mathcal{M})$, a <u>measure</u> is a function $\mu : \mathcal{M} \to [0,+\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ disjoint,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a <u>measure space</u>.

<u>Thm</u>: For any measure space $(X, \mathcal{M}, \mu)$ and $A, B \in \mathcal{M}$, $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$,

(i) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

(ii) $A \subseteq B$, $\mu(A) < +\infty$
$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$$

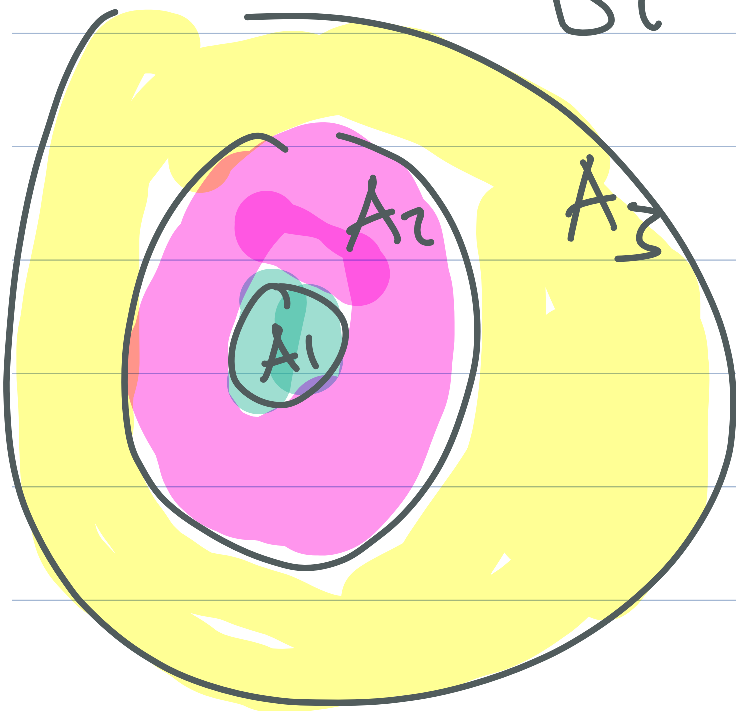
(iii) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$

(iv) $A_i \subseteq A_{i+1} \ \forall i \Rightarrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i)$
cty from below

(v) $A_{i+1} \subseteq A_i \ \forall i$, $\mu(A_1) < +\infty$ cty from above
$$\Rightarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i)$$

Pf: Last time, we showed (i)-(iii).

(iv) Define $B_1 = A_1$,
$B_i = A_i \setminus A_{i-1}$ for $i \geq 2$.



By def, $\bigcup_{i=1}^{n} B_i = A_n$.

Hence, $\bigcup_{i=1}^{\infty} B_i = \bigcup_{n=1}^{\infty} A_n$.

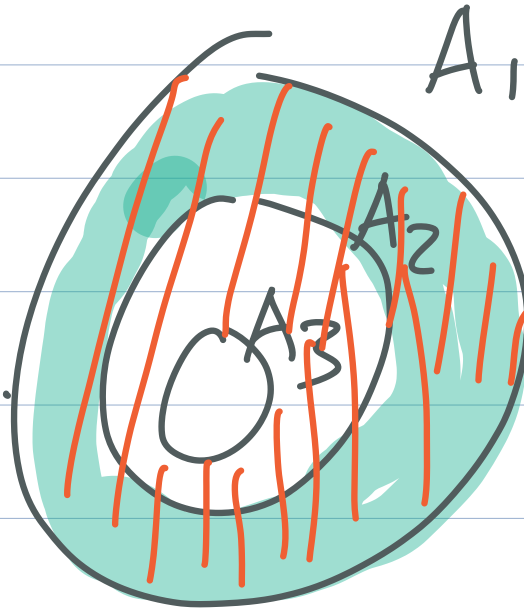
Then $\mu(A_n) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i)$.
Consequently,

$$\lim_{n\to\infty} \mu(A_n) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

(v)

Define $B_i = A_1 \setminus A_i$



By constr., $B_i \subseteq B_{i+1}$.

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_1 \setminus A_i)$$

$$= \bigcup_{i=1}^{\infty} (A_1 \cap A_i^c) = A_1 \cap \left(\bigcup_{i=1}^{\infty} A_i^c \right)$$

$$= A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i \right)^c = A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i \right)$$

Therefore,

$$\mu(A_1) = \mu\Big(\overbrace{A_1 \cap \big(\bigcap_{i=1}^{\infty} A_i\big)}^{= \bigcap_{i=1}^{\infty} A_i}\Big) + \mu\Big(\underbrace{A_1 \setminus \bigcap_{i=1}^{\infty} A_i}_{\bigcup_{i=1}^{\infty} B_i}\Big)$$

$$= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \lim_{i\to\infty} \mu(B_i)$$ by cty from below

$$= \mu\left(\bigcap_{i=1}^{\infty} A_i\right) + \underbrace{\lim_{i\to\infty} \mu(A_1) - \mu(A_i)}_{= \mu(A_1) - \lim_{i\to\infty} \mu(A_i)}.$$

Rearranging gives

$$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right). \qquad \square$$

Ex: (Why $\mu(A_1) < +\infty$ is necessary for cty from above.)

$X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$, $\mu(E) = |E|$.

Let $A_i = \{n \in \mathbb{N} : n \geq i\}$

$\bigcap_{i=1}^{\infty} A_i = \emptyset$

Then $\mu\left(\bigcap_{i=1}^{\infty} A_i\right)=0$.

However $\lim_{i\to\infty}\mu(A_i)=+\infty$.

Measure Terminology

$(X, \mathcal{M}, \mu)$

- μ is finite measure if $\mu(X)<+\infty$.
- μ is a σ-finite measure if $\exists\, \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i)<+\infty\ \forall i \in \mathbb{N}$.
- $E \in 2^X$ is a null set (of μ) if $E \in \mathcal{M}$ and $\mu(E)=0$.
- We say that a property holds (μ)-almost every $x \in X$ if the set of points where it fails is a null set.

abbreviate: μ-a.e or a.e.

Recall initial goal:
Find a measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, where $\mu((a,b)) = b-a$ and μ is translation invariant.

In order to prove such a measure exists... give up countable additivity; can apply to all 2^X

Outer Measures

Def: An <u>outer measure</u> on X is a function $\mu^*: 2^X \to [0, +\infty]$ s.t.
(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*(\underbrace{\bigcup_{i=1}^{\infty} A_i}_{\text{not necessarily disjoint}}) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Rmk:

(ii)+(iii) $\Leftrightarrow$ If $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Ex: (Lebesgue outer measure)

Define $\mu^*: 2^{\mathbb{R}} \to [0, +\infty]$ by

$$\mu^*(A) = \inf \underbrace{\left\{ \sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}}_{S}$$

$a_i \leq b_i$

- We will prove μ^* is an outer measure.
- $\mu^*((a,b]) = b-a$, $a \leq b$.

To see this...

- $a_1 = a$, $b_1 = b$, $a_i = b_i = 0$ $\forall i \geq 2$.
  Then $b-a \in S$, so
  $\inf(S) \leq b-a$.
- Since any choice $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ must satisfy

$$[a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$$

  the total length of the covering must be at least $b-a$. Thus $b-a$ is a lower bound for S and $\inf(S) \geq b-a$.

  will rigorously justify soon...

- μ^* is translation invariant
- Is it countable additive? No!

But: we will show that it becomes countably additive when restricted to "nice enough" sets...

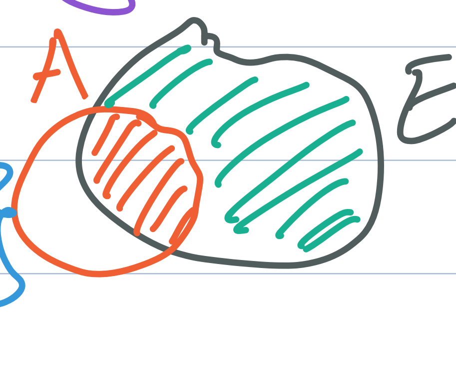
... which sets are "nice measure"?

Def: Given any outer measure μ^*, $A \subseteq X$ is <u>μ^*-measurable</u> if

(*) $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subseteq X$

"A breaks apart any set nicely"

Let $\mathcal{M}_{\mu^*} = \{A \subseteq X : A \text{ is } \mu^*\text{-meas}\}$



Rmk: By countable subadditivity, $\leq$ holds in (*) for all $A \subseteq X$.
Thus, to show $A \in \mathcal{M}_{\mu^*}$, it suffices to show $\geq$ in (*).

Prop: For any outer measure μ^*, if $\mu^*(B)=0$, then $B \in \mathcal{M}_{\mu^*}$.

Pf: For all $E \subseteq X$, by monotonicity,

$$\mu^*(E) \geq 0 + \mu^*(E \cap B^c)$$
$$= \mu^*(E \cap B) + \mu^*(E \cap B^c) \quad \square$$

Thm (Carathéodory): Given an outer measure μ^*,

(i) $\mathcal{M}_{\mu^*}$ is a σ-algebra
(ii) μ^* is a measure on $\mathcal{M}_{\mu^*}$.

Q: Is this the "largest" σ-algebra on which μ^* is a measure?
A: No ⍨. See HW3.

<u>Prop</u>: $\mathcal{M}_{\mu^*}$ is an algebra and μ^* is finitely additive on $\mathcal{M}_{\mu^*}$.

<u>Pf</u>:
Since $\mu^*(\emptyset)=0 \overset{\text{Prop}}{\Longrightarrow} \emptyset \in \mathcal{M}_{\mu^*}$, hence $\mathcal{M}_{\mu^*}$ is nonempty.
... finish next time ☺