# Lecture 4

HW2 (8)(b)

Hint: Can you show that

$f_*(x) < f^*(x)$

$\iff \exists\, p, q \in \mathbb{Q}$ s.t.

$$f_*(x) \leq p < q \leq f^*(x)$$

Recall:

# Measure Terminology

$(X, \mathcal{M}, \mu)$

- $\mu$ is finite measure if $\mu(X) < +\infty$.
- $\mu$ is a $\sigma$-finite measure if $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty \ \forall i \in \mathbb{N}$.
- $E \in 2^X$ is a null set (of $\mu$) if $E \in \mathcal{M}$ and $\mu(E) = 0$.
- We say that a property holds for ($\mu$)-almost every $x \in X$ if the set of points where it fails is a null set.

abbreviate: $\mu$-a.e or a.e.

# Outer Measures

**Def**: An outer measure on $X$ is a function $\mu^*: 2^X \to [0, +\infty]$ s.t.

(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*\left(\underbrace{\bigcup_{i=1}^{\infty} A_i}_{\text{not necessarily disjoint}}\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

**Def**: Given any outer measure $\mu^*$, $A \subseteq X$ is $\mu^*$-measurable if

(*) $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, $\forall E \subseteq X$

"A breaks apart any set nicely"

Let $\mathcal{M}_{\mu^*} = \{A \subseteq X : A \text{ is } \mu^*\text{-meas}\}$

Prop: For any outer measure $\mu^*$, if $\mu^*(B)=0$, then $B \in \mathcal{M}_{\mu^*}$.

Thm (Carathéodory): Given an outer measure $\mu^*$,
(i) $\mathcal{M}_{\mu^*}$ is a $\sigma$-algebra
(ii) $\mu^*$ is a measure on $\mathcal{M}_{\mu^*}$.

Before proving this theorem, we will prove the following Prop

Prop: $\mathcal{M}_{\mu^*}$ is an algebra and $\mu^*$ is finitely additive on $\mathcal{M}_{\mu^*}$.

Furthermore, given $\{B_i\}_{i=1}^{n} \subseteq \mathcal{M}_{\mu^*}$ disjoint,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) = \sum_{i=1}^{n} \mu^*(E \cap B_i), \ \forall E \subseteq X.$$

(*)

Pf:

Since $\mu^*(\emptyset) = 0 \overset{\text{Prop}}{\Longrightarrow} \emptyset \in \mathcal{M}_{\mu^*}$, hence $\mathcal{M}_{\mu^*}$ is nonempty.

Furthermore, $\mathcal{M}_{\mu^*}$ is closed under complements by definition of $\mathcal{M}_{\mu^*}$ (since defn is symmetric in $A$ vs $A^c$).

To see that $\mathcal{M}_{\mu^*}$ is closed under finite unions, it suffices to show

$$A, B \in \mathcal{M}_{\mu^*} \Rightarrow A \cup B \in \mathcal{M}_{\mu^*}.$$

Suppose $A, B \in \mathcal{M}_{\mu^*}$. Fix $E \subseteq X$.

$$\begin{aligned}
&\mu^*(E) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap \overbrace{A^c \cap B^c}^{(A \cup B)^c})
\end{aligned}$$

$$\geq \underbrace{\mu^*\big(\underbrace{(E\cap A)\cup(E\cap A^c\cap B)}_{E\cap(A\cup B)}\big)}_{\text{subadditivity}} + \mu^*(E\cap(A\cup B)^c)$$

$$= \mu^*(E\cap(A\cup B)) + \mu^*(E\cap(A\cup B)^c)$$

Therefore, $A\cup B \in \mathcal{M}_{\mu^*}$.

Hence $\mathcal{M}_{\mu^*}$ is an algebra.

To show the second property, suppose $\{B_i\}_{i=1}^{n} \subseteq \mathcal{M}_{\mu^*}$ disjoint. Fix $E\subseteq X$.

Base case: $n=1$ ✓

Inductive step: assume $(*)$ for $n-1$.

$$\mu^*\left(E\cap\left(\bigcup_{i=1}^{n} B_i\right)\right)$$

since $B_n \in \mathcal{M}_{\mu^*}$

$$= \mu^*\left(E\cap\left(\bigcup_{i=1}^{n} B_i\right)\cap B_n\right) + \mu^*\left(E\cap\left(\bigcup_{i=1}^{n} B_i\right)\cap B_n^c\right)$$

disjoint

$$= \mu^*(E\cap B_n) + \mu^*\left(E\cap\left(\bigcup_{i=1}^{n-1} B_i\right)\right)$$

ind. hyp.

$$= \mu^*(E\cap B_n) + \sum_{i=1}^{n-1} \mu^*(E\cap B_i)$$

$$= \sum_{i=1}^{n} \mu^*(E\cap B_i)$$

Therefore (*) holds.
Taking $E = X$, this shows
$\mu^*$ is finitely additive
on $\mathcal{M}_{\mu^*}$. $\square$

Prop: Given $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_{\mu^*}$ disjoint, for all $E \subseteq X$,

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Pf:

Note that "$\leq$" follows by subadditivity. It remains to show "$\geq$".

Since $\mathcal{M}_{\mu^*}$ is closed under finite unions, $\bigcup_{i=1}^{n} B_i \in \mathcal{M}_{\mu^*}$. Thus,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{n} B_i\right)^c\right)$$

↓ previous Prop ↓ monotonicity

$$\geq \sum_{i=1}^{n} \mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Sending $n \to \infty$ gives the result. $\square$

Proof of Caratheodory's Theorem:

To show $\mathcal{M}_{\mu^*}$ is a $\sigma$-algebra, it suffices to show it is closed under countable disjoint unions.

Given $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_{\mu^*}$ disjoint, we must show $\bigcup_{i=1}^{\infty} B_i \in \mathcal{M}_{\mu^*}$.

Fix $E \subseteq X$. By Prop,

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*\Big(E \cap \Big(\bigcup_{i=1}^{\infty} B_i\Big)^c\Big)$$

$\downarrow$ subadditivity

$$\geq \mu^*\Big(E \cap \Big(\bigcup_{i=1}^{\infty} B_i\Big)\Big) + \mu^*\Big(E \cap \Big(\bigcup_{i=1}^{\infty} B_i\Big)^c\Big)$$

Thus $\bigcup_{i=1}^{\infty} B_i \in \mathcal{M}_{\mu^*}$.

Finally, $\mu^*|_{\mathcal{M}_{\mu^*}}$ is countably additive by taking $E = \bigcup_{i=1}^{\infty} B_i$ in previous proposition. $\square$

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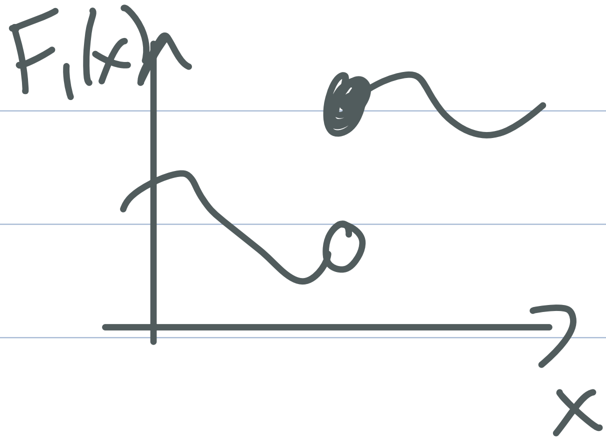
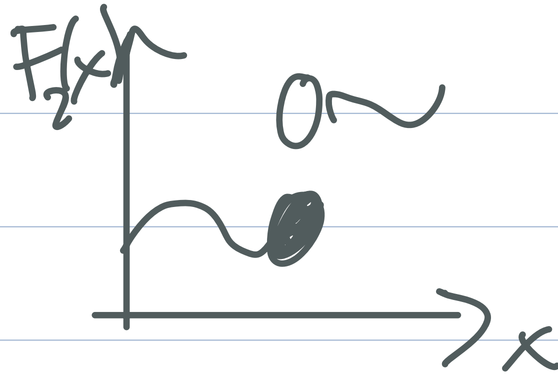
Back to Lebesgue outer measure... in fact, generalize to Lebesgue-Stieltjes measures...

Recall: $F: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is <u>right continuous</u> if $\forall x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(y) = F(x).$$




<u>Def</u>: Given $F:\mathbb{R}\to\mathbb{R}$ nondecreasing and right continuous, define $\mu_F^*: 2^{\mathbb{R}} \to [0,+\infty]$ by

$$\mu_F^*(A) := \inf\left\{ \overbrace{\sum_{i=1}^{\infty} F(b_i)-F(a_i)}^{\text{abbreviate } |I_i|_F} : A \subseteq \bigcup_{i=1}^{\infty} \underbrace{(a_i,b_i]}_{I_i},\ a_i\le b_i \right\}$$

Why do we require F nondecreasing and right continuous?

Spoiler: We will show that

any finite measure $\mu$ on $\mathcal{B}_{\mathbb{R}}$ satisfies $\mu = \mu_F^* |_{\mathcal{B}_{\mathbb{R}}}$

for $F(x) = \mu((-\infty, x])$.

This is known as the <u>cumulative distribution function</u> of $\mu$.

Note that if $\mu$ is a finite measure on $\mathcal{B}_{\mathbb{R}}$ and $F(x)$ is its CDF...

- $F$ is nondecreasing: if $x \leq y$
  $F(x) = \mu((-\infty, x]) \leq \mu((-\infty, y]) = F(y)$.
- $F$ is right continuous:

for any sequence $x_n \downarrow x$,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n])$$

cty from above

$$= \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right)$$

$$= \mu((-\infty, x])$$

$$= F(x).$$

**Thm:** For any $F: \mathbb{R} \to \mathbb{R}$ nondecreasing and right cts, $\mu^*_F$ is an outer measure.

Pf:

Since $\mu^*_F \geq 0$ always and

$$\emptyset \subseteq \bigcup_{i=1}^{\infty} (1,1],$$

$$\mu_F^*(\emptyset) \le \sum_{i=1}^{\infty} F(1) - F(1) = 0,$$

we conclude $\mu_F^*(\emptyset) = 0$.

It remains to show

$$A \subseteq \bigcup_{j=1}^{\infty} B_j \Rightarrow \mu_F^*(A) \le \sum_{j=1}^{\infty} \mu_F^*(B_j).$$

Fix $A \subseteq \bigcup_{j=1}^{\infty} B_j$.

WLOG $\sum_{j=1}^{\infty} \mu_F^*(B_j) < +\infty$, so

$\mu_F^*(B_j) < +\infty \quad \forall j \in \mathbb{N}.$

Thus, by defn of inf, $\forall \varepsilon>0, j\in\mathbb{N}$,
$\exists \{I_i^{j,\varepsilon}\}_{i=1}^{\infty}$ s.t.

- $B_j \subseteq \bigcup_{i=1}^{\infty} I_i^{j,\varepsilon}$

- $\mu_F^*(B_j) \leq \sum_{i=1}^{\infty} |I_i^{j,\varepsilon}|_F \leq \mu_F^*(B_j) + \frac{\varepsilon}{2^j}$

Since $A \subseteq \bigcup_{j=1}^{\infty} B_j$, $A \subseteq \bigcup_{i,j=1}^{\infty} I_i^{j,\varepsilon}$

$$\mu_F^*(A) \leq \sum_{i,j=1}^{\infty} |I_i^{j,\varepsilon}|_F$$
$$\leq \sum_{j=1}^{\infty} \mu_F^*(B_j) + \frac{\varepsilon}{2^j}$$
$$= \sum_{j=1}^{\infty} \mu_F^*(B_j) + \varepsilon$$

Sending $\varepsilon \to 0$ gives the result. $\square$

**Thm:** For all $a, b \in \mathbb{R}$, $a \leq b$,
$\mu_F^*((a,b]) = F(b) - F(a)$.

Pf: "$\leq$" follows quickly, since
$(a,b] \subseteq (a,b] \cup (1,1] \cup (1,1] \cup \dots$
$\mu_F^*((a,b]) \leq F(b) - F(a) + 0 + 0 + \dots$
$= F(b) - F(a)$.

Now, we show "$\geq$".
WLOG, assume $a < b$.
It suffices to show that $F(b) - F(a)$ is a lower bound for the set

$$\left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : (a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], a_i \leq b_i \right\}$$

... next time :)