Lecture 5

<u>Thm</u> (Carathéodory): Given an outer measure μ^*,

(i) $\mathcal{M}_{\mu^*}$ is a σ-algebra

(ii) μ^* is a measure on $\mathcal{M}_{\mu^*}$.

Recall: $F : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is <u>right continuous</u> if $\forall x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(y) = F(x).$$

Def: Given $F:\mathbb{R}\to\mathbb{R}$ nondecreasing and right continuous, define $\mu_F^*: 2^{\mathbb{R}} \to [0,+\infty]$ by

$$\mu_F^*(A) := \inf\left\{ \sum_{i=1}^{\infty} F(b_i)-F(a_i) : A \subseteq \bigcup_{i=1}^{\infty} (a_i,b_i], a_i \leq b_i \right\}$$

Lebesgue-Stieltjes outer measure

Rmk: (Optimization terminology)
Given $f: X \to \mathbb{R}\cup\{+\infty\}$, $C \subseteq X$, $C \neq \emptyset$, consider the optimization problem

$$m := \inf_{x\in C} f(x) = \inf\{f(x) : x \in C\}$$

C is the "constraint set"
f is the "objective function"

m is the "optimum"

any sequence $\{x_n\}_{n=1}^{\infty} \subseteq C$ s.t. $\lim_{n\to\infty} f(x_n) = m$ is a "minimizing sequence"

There may exist...

$x \in \{x : f(x) < +\infty\} \cap C$ is a "feasible point"

$x \in C$ s.t. $f(x) = m$ is an "optimizer"/ "point that attains the optimum"

if an optimizer x exists, m is the "minimum" and x is the "minimizer"

Thm: For any $F: \mathbb{R} \to \mathbb{R}$ nondecreasing and right cts, μ^*_F is an outer measure.

Thm: For all $a, b \in \mathbb{R}$, $a \leq b$,
$\mu_F^*((a,b]) = F(b) - F(a)$

Pf:
Last time: "$\leq$" ✓

Now, we show "$\geq$".
WLOG, assume $a < b$.
It suffices to show that $F(b) - F(a)$ is a lower bound for the set

$$\left\{ \sum_{i=1}^{\infty} F(b_i) - F(a_i) : (a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i], a_i \leq b_i \right\}$$

We will show that
$$(a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$$
$$\Rightarrow F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i).$$

Fix $(a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$ and $\varepsilon > 0$.

Since F is right continuous, $\exists\, \delta_i > 0$ s.t.

$$F(b_i + \delta_i) < F(b_i) + \frac{\varepsilon}{2^i}$$

Furthermore,
$$[a+\varepsilon, b] \subseteq (a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \delta_i)$$

Since $[a+\varepsilon, b]$ is cpt, $\exists$ finite subcover s.t.

$$[a+\varepsilon, b] \subseteq \bigcup_{i=1}^{N} (a_i, b_i+\delta_i)$$

WLOG...

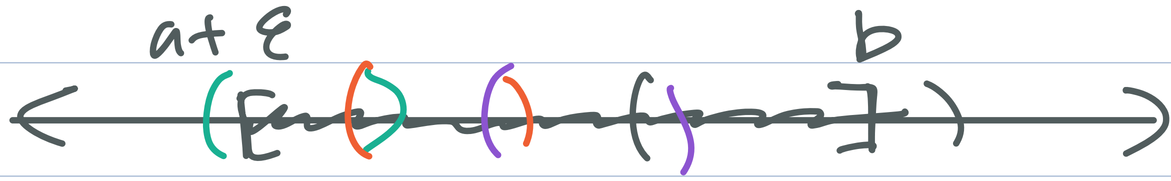
(i) we may discard any open $(a_i, b_i+\delta_i)$ interval s.t. $(a_i, b_i+\delta_i) \subseteq (a_j, b_j+\delta_j)$ for $j \neq i$. This ensures $a_i < b_i+\delta_i$ $\forall i$.

(ii) we may discard any $(a_i, b_i+\delta_i)$ s.t. $a_i \geq b$ or $b_i+\delta_i \leq a+\varepsilon$.

(iii) we may index intervals s.t. $b_i+\delta_i < b_{i+1}+\delta_{i+1}$ $\forall i$.

(We never will have $b_i+\delta_i = b_{i+1}+\delta_{i+1}$, or we would have discarded it.)

CLAIM: $b_i+\delta_i > a_{i+1}$ $\forall i=1,\dots,n-1$



Pf of Claim: Assume, for the sake of contr, that $a_{i_0+1} \geq b_{i_0}+\delta_{i_0}$ for some i_0. Then $a_{i_0+1} \notin (a_{i_0}, b_{i_0}+\delta_{i_0}) \cup (a_{i_0+1}, b_{i_0+1}+\delta_{i_0+1})$. By (iii), $a_{i_0+1} \notin (a_j, b_j+\delta_j)$ for $j<i_0$. By (i), we can't have $a_{i_0+1} \in (a_j, b_j+\delta_j)$ for $j>i_0+1$ or else we'd have $a_j < a_{i_0+1} < b_{i_0+1}+\delta_{i_0+1} < b_j+\delta_j$. Thus $a_{i_0+1} \notin \bigcup_{j=1} (a_j, b_j+\delta_j)$.

By (ii), $b_i+\delta_i > a+\varepsilon$ $\forall i=1,\dots,n$, so $a_{i_0+1} \geq b_{i_0}+\delta_{i_0} > a+\varepsilon$. Likewise $b < a_{i_0+1}$. Thus, $a_{i_0+1} \in [a+\varepsilon, b]$. This shows $[a+\varepsilon, b] \not\subseteq \bigcup_{j=1}$, which is a contradiction.

Since F is nondecreasing,

$$
\begin{aligned}
F(b) - F(a+\varepsilon) &\le F(b_N+\delta_N) - F(a_1) \\
&= F(b_N+\delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(a_{i+1}) - F(a_i)
\end{aligned}
$$

CLAIM: $a_{i+1} < b_i+\delta_i$

$$
\begin{aligned}
&\le F(b_N+\delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(b_i+\delta_i) - F(a_i) \\
&\le \sum_{i=1}^{N} \overbrace{F(b_i) - F(a_i)}^{\text{nonneg}} + \frac{\varepsilon}{2^i} \\
&\le \left(\sum_{i=1}^{\infty} F(b_i) - F(a_i) \right) + \varepsilon .
\end{aligned}
$$

Sending $\varepsilon \to 0$ gives the result.

<u>Def</u>: (Lebesgue measure/outer measure)

When $F(x) = x$, we write

$\lambda^* := \mu_F^* =$ Lebesgue outer measure

$\mathcal{M}_{\lambda^*} := \mathcal{M}_{\mu_F^*} =$ Lebesgue measurable sets

$\lambda = \lambda^*|_{\mathcal{M}_{\lambda^*}} =$ Lebesgue measure

<u>Def</u>: (Lebesgue-Stieltjes measure):

For any $F: \mathbb{R} \to \mathbb{R}$ nondec, right cts,

$\mu_F := \mu_F^*|_{\mathcal{M}_{\mu_F^*}}$

Is μ_F a Borel measure?

That is, do we have $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F^*}$?

Thm:

For any $F:\mathbb{R}\to\mathbb{R}$ nondec, right cts,
$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F^*}$.

Pf: It suffices to show
$(-\infty, b] \in \mathcal{M}_{\mu_F^*}$ $\forall\, b\in\mathbb{R}$,
that is, we must show
$\mu_F^*(E) = \mu_F^*(E\cap(-\infty,b]) + \mu_F^*(E\cap(-\infty,b]^c)$
for any $E\subseteq\mathbb{R}$, $b\in\mathbb{R}$.

WLOG, we may assume $\mu_F^*(E)<+\infty$.
Fix $\varepsilon>0$. $\exists$ $\{(a_i,b_i]\}_{i=1}^{\infty}$
s.t. $E\subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]$ and
$$\sum_{i=1}^{\infty} F(b_i)-F(a_i) \le \mu_F^*(E)+\varepsilon.$$

Note that

$$(a_i, b_i] \cap (-\infty, b] \subseteq (a_i, \min\{b, b_i\}]$$

$$(a_i, b_i] \cap (b, +\infty) \subseteq [\max\{a_i, b\}, b_i]$$

So,

$$E \cap (-\infty, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, \min\{b, b_i\}]$$

$$E \cap (b, +\infty) \subseteq \bigcup_{i=1}^{\infty} [\max\{a_i, b\}, b_i]$$

$(s)_+ = \max\{s, 0\}$

Thus,

$$\mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c)$$

$$\leq \sum_{i=1}^{\infty} \underbrace{\left(F(\min\{b_i, b\}) - F(a_i)\right)_+}_{\mathrm{I}_i} + \sum_{i=1}^{\infty} \underbrace{\left(F(b_i) - F(\max\{a_i, b\})\right)_+}_{\mathrm{II}_i}$$

$= (*)$

Consider the i^{th} term of sum:

(i) If $b \le a_i$, $\mathrm{I}_i = 0$, $\mathrm{II}_i = F(b_i) - F(a_i)$

(ii) If $b > b_i$, $\mathrm{II}_i = 0$, $\mathrm{I}_i = F(b_i) - F(a_i)$

(iii) If $b \in (a_i, b_i]$, then

$$\mathrm{I}_i + \mathrm{II}_i = F(\cancel{b}) - F(a_i) + F(b_i) - \cancel{F(b)}$$
$$= F(b_i) - F(a_i)$$

Hence

$$(*) = \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

$$\le \mu^*_F(E) + \varepsilon$$

Sending $\varepsilon \to 0$ gives the result. □

The last item we hoped to have for Lebesgue measure is...

Thm: λ^* is translation invariant on $2^{\mathbb{R}}$; λ is translation invariant on $\mathcal{M}_{\lambda^*}$.

Pf: For any $a \in \mathbb{R}$, $A \subseteq \mathbb{R}$,

$$A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \iff A+a \subseteq \bigcup_{i=1}^{\infty} (a_i+a, b_i+a].$$

and both covers have the same length since $F(x)=x$ implies

$$F(b_i+a) - F(a_i+a) = F(b_i) - F(a_i).$$

Thus $\lambda^*(A) = \lambda^*(A+a)$. ∎

Since $\lambda = \lambda^*|_{\mathfrak{M}_{\lambda^*}}$, it suffices

to show $A \in \mathfrak{M}_{\lambda^*} \Rightarrow A + a \in \mathfrak{M}_{\lambda^*}$ $\forall\, a \in \mathbb{R}$. If this holds, then $\forall\, A \in \mathfrak{M}_{\lambda^*}, a \in \mathbb{R}$,
$\lambda(A) = \lambda^*(A) = \lambda^*(A+a) = \lambda(A+a)$.

Fix $E \subseteq \mathbb{R}$, $A \in \mathfrak{M}_{\lambda^*}$, $a \in \mathbb{R}$.
WTS $\lambda^*(E) \geq \lambda^*(E \cap (A+a)) + \lambda^*(E \cap (A+a)^c)$

finish next time :)