Lecture 7

<u>Thm</u>: Suppose μ is a finite Borel measure on $\mathbb{R}$. Define $F:\mathbb{R}\to\mathbb{R}$ by
$$F(x) := \mu((-\infty, x]).$$
Then
(i) F is nondecreasing, right cts
(ii) $\mu = \mu_F$.

<u>Lemma</u>: Given $F:\mathbb{R}\to\mathbb{R}$ nondecreasing, right cts, $\forall E \in \mathcal{M}_{\mu_F^*}$,

$$\mu_F(E) = \inf\left\{\sum_{i=1}^{\infty} \mu_F((a_i, b_i]) : E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]\right\}$$ $a_i \le b_i$

Thm: (Regularity of L-S measures)
For any $E \in \mathcal{M}_{\mu_F}$,

"outer regular"

$$\mu_F(E) \overset{(*)}{=} \inf\{\mu_F(\mathcal{U}) : E \subseteq \mathcal{U},\ \mathcal{U} \text{ open}\}$$

"inner regular"

$$\overset{(**)}{=} \sup\{\mu_F(K) : K \subseteq E,\ K \text{ cpt}\}$$

This completes our study of measure (for now).

On to integration...

Measurable Functions

Given $f: X \to Y$, $f^{-1}(E) = \{x \in X : f(x) \in E\}$

<u>Recall from HW1:</u>

$f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$, $f^{-1}(E^c) = f^{-1}(E)^c$

$f^{-1}\left(\bigcap_\alpha E_\alpha\right) = \bigcap_\alpha f^{-1}(E_\alpha)$.

Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces, then the following are σ-algebras

$\{f^{-1}(E) : E \in \mathcal{N}\}$ "pullback of $\mathcal{N}$"
$\{E : f^{-1}(E) \in \mathcal{M}\}$ "pushforward of $\mathcal{M}$"

Def: $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if, for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.

"the preimage of every measurable set is measurable"

$\Updownarrow$

$$\{f^{-1}(E) : E \in \mathcal{N}\} \subseteq \mathcal{M}$$

$\Updownarrow$

$$\{E : f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{N}$$

$f: X \to \overline{\mathbb{R}}$

Whenever $f: X \to \mathbb{R}$, we will suppose the range is endowed with $\mathcal{B}_{\mathbb{R}}$.

$\mathcal{B}_{\overline{\mathbb{R}}}$

Def: $f: \mathbb{R} \to \mathbb{R}$ is <u>Lebesgue measurable</u> if it is $(\mathcal{M}_{\lambda^*}, \mathcal{B}_{\mathbb{R}})$-measurable.

Given X, Y topological spaces, $f: X \to Y$ is Borel measurable if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Rmk: Suppose $f: \mathbb{R} \to \mathbb{R}$

f Lebesgue meas $\Leftarrow$ f Borel meas

... since we will see $\mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{M}_{\lambda^*}$.

Prop: Given measurable spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$, where $\mathcal{N}$ is generated by $\mathcal{E}$. Then $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff, $\forall E \in \mathcal{E}$, $f^{-1}(E) \in \mathcal{M}$.

Pf: "$\Rightarrow$" is immediate since $\mathcal{E} \subseteq \mathcal{N}$.

Now, show "$\Leftarrow$".

this is a σ-algebra of subsets of Y

Note that
$$\mathcal{E} \subseteq \{E : f^{-1}(E) \in \mathcal{M}\}$$

Since $\mathcal{E}$ generates $\mathcal{N}$, this implies
$$\mathcal{N} \subseteq \{E : f^{-1}(E) \in \mathcal{M}\}. \qquad \square$$

Cor: If X and Y are topological spaces, then every continuous function $f: X \to Y$ is Borel meas.

Pf: For every open set $\mathcal{U} \subseteq Y$, $f^{-1}(\mathcal{U})$ is open, hence $f^{-1}(\mathcal{U}) \in \mathcal{B}_X$. Since open sets generate $\mathcal{B}_Y$, this completes the proof. $\square$

Cor: If $(X, \mathcal{M})$ is a measurable space and $f: X \to \overline{\mathbb{R}}$, then

$$f \text{ is } (\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})\text{-measurable}$$
$$\Updownarrow$$
$$f^{-1}([(a, +\infty]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Pf: On HW3, you show that $\{[a, +\infty] : a \in \mathbb{R}\}$ generates $\mathcal{B}_{\overline{\mathbb{R}}}$.

Thus this is an immediate consequence of Prop. $\square$

Question: If $(X, \mathcal{M})$ is a measurable space and $f : X \to \overline{\mathbb{R}}$ is given by
$$f(x) = c \quad \forall x \in X$$
for $c \in \overline{\mathbb{R}}$, is f $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-meas?

Answer: Yes!

Thm: Fix a measurable space $(X, \mathcal{M})$ and $\{f_i\}_{i=1}^{\infty}$, $f_i : X \to \overline{\mathbb{R}}$, that are $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable. Then the following are also $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

CONVENTIONS

(i) $f_1 + f_2$ $\leftarrow$ $-\infty + (+\infty) = 0$

(ii) $f_1 \cdot f_2$ $\leftarrow$ $0 \cdot (\pm\infty) = 0$

(iii) $f_1 \vee f_2$ $\quad f_1 \vee f_2(x) := \max\{f_1(x), f_2(x)\}$

(iv) $f_1 \wedge f_2$ $\quad f_1 \wedge f_2(x) := \min\{f_1(x), f_2(x)\}$

(v) $\sup_i f_i$

(vi) $\inf_i f_i$

(vii) $\limsup_{i \to \infty} f_i$

(viii) $\liminf_{i \to \infty} f_i$

(ix) $\lim_{i \to \infty} f_i$, as long as $\lim_{i \to \infty} f_i(x)$ exists $\forall x \in X$.

Pf:

(i) + (ii): HW4

(iii) $(f_1 \vee f_2)^{-1}((a,+\infty])$
$= \{x \in X : f_1 \vee f_2(x) > a\}$
$= \{x \in X : f_1(x) > a\} \cup \{x \in X : f_2(x) > a\}$
$= f_1^{-1}((a,+\infty]) \cup f_2^{-1}((a,+\infty]) \in \mathcal{M}$

(iv) Reduce to (iii), since
$f_1 \wedge f_2 = -((-f_1) \vee (-f_2))$

(v) $(\sup_i f_i)^{-1}((a,+\infty])$
$= \bigcup_{i=1}^{\infty} f_i^{-1}((a,+\infty])$

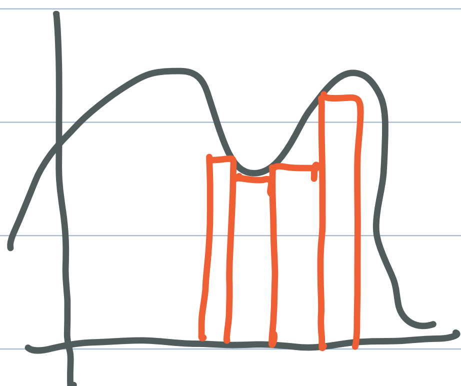
(vi)-(ix) follow from what has been shown.

Rmk: Consider $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

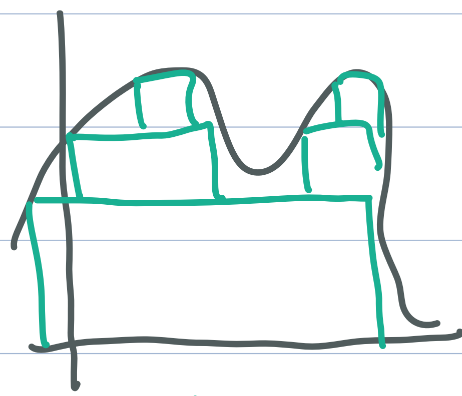
f Borel meas
g Borel meas $\Rightarrow$ $f \circ g$ Borel meas

f Borel meas
g Lebesgue meas $\Rightarrow$ $f \circ g$ Lebesgue meas

$(f \circ g)^{-1}(E) = g^{-1}(f^{-1}(E))$



Riemann

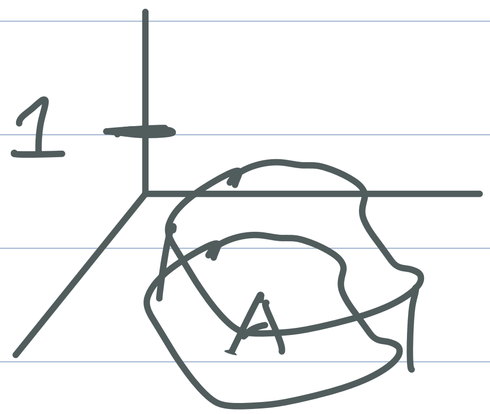


Lebegue

Simple functions

Def: For any $A \subseteq X$, the indicator function of A is

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$



When is an indicator fn meas?

$$\mathbb{1}_A^{-1}(B) = \begin{cases} A & \text{if } 1 \in B,\ 0 \notin B \\ A^c & 0 \in B,\ 1 \notin B \\ \emptyset & 0, 1 \notin B \\ X & 0, 1 \in B \end{cases}$$

Thus, if $A \in \mathcal{M}$, $1_A^{-1}(B) \in \mathcal{M}$ $\forall B \subseteq \mathbb{R}$.

<u>Def</u>: A $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable function $f: X \to \mathbb{R}$ is a <u>simple function</u> if $f(X)$ is a finite subset of $\mathbb{R}$.

The standard representation is

$$f(x) = \sum_{i=1}^{n} c_i \, 1_{E_i}(x)$$

for $f(X) = \{c_1, \dots, c_n\}$ distinct

$E_i = f^{-1}(c_i)$

<u>Rmk</u>: $\{E_i\}_{i=1}^{n}$ form a disjoint partition of X.

Rmk: Consider $f(x)=2$, $X=\mathbb{R}$

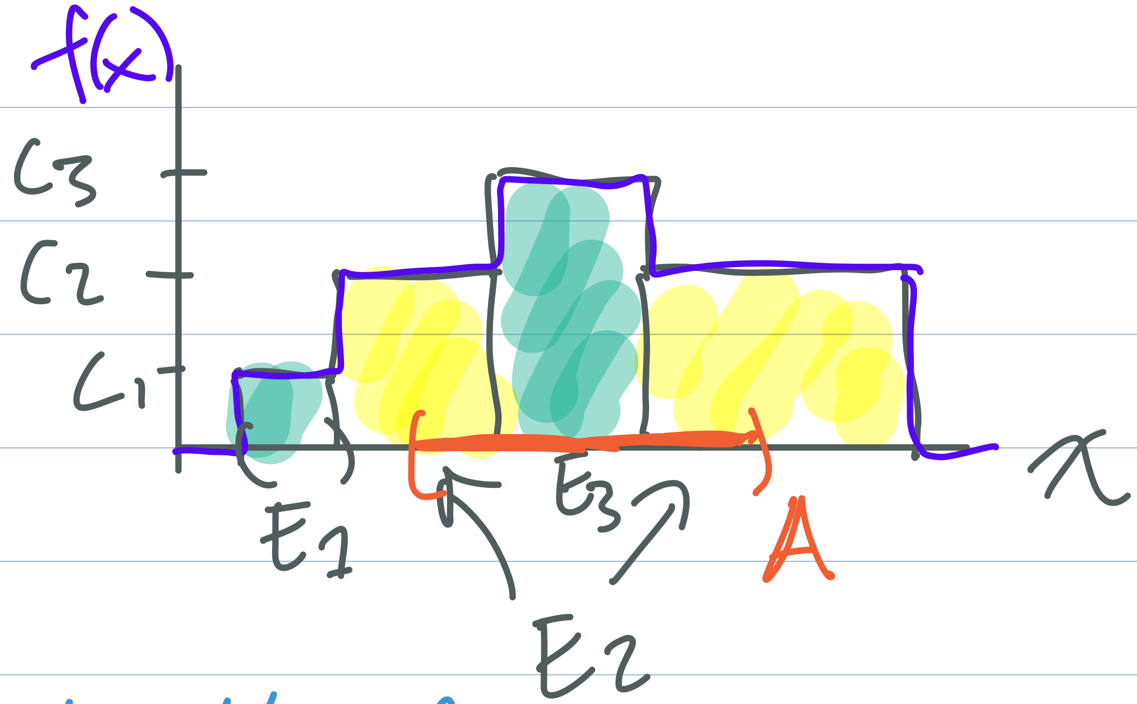
$$f(x)=2\cdot 1_{\mathbb{R}}(x)=2\cdot 1_{(-\infty,0]}(x)+2\cdot 1_{(0,+\infty)}(x)$$

Def: For any measure space $(X,\mathcal{M},\mu)$ and nonnegative simple function $f(x)$ with standard representation

$$f(x)=\sum_{i=1}^{n} c_i 1_{E_i}(x)$$

we define

$$\int f\,d\mu := \sum_{i=1}^{n} c_i\,\mu(E_i)$$



For $A \in \mathcal{M}$, define

$$\int_A f\,d\mu := \int \underbrace{f \cdot \mathbb{1}_A}_{\text{nonnegative, simple fn}} d\mu$$

Rmk (Notation):

$$\int f\,d\mu = \int f(x)\,d\mu(x) = \int_X f(x)\,d\mu(x)$$

$$\int_A f\,d\mu = \int_A f(x)\,d\mu(x)$$

Prop: Given a measure space $(X, \mathcal{M}, \mu)$ and nonnegative simple fns $f, g : X \to \mathbb{R}$,

① if $c \geq 0$, $\int cf\,d\mu = c\int f\,d\mu$

② $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

③ $f \leq g \Rightarrow \int f\,d\mu \leq \int g\,d\mu$

④ The function

$$A \mapsto \int_A f\,d\mu$$

is a measure on $(X, \mathcal{M})$.

= END OF MATERIAL FOR MID 1 =