Lecture 9

Recall:

Integrate nonneg, meas fns

Thm: Given $f: X \to [0,+\infty]$ measurable, there exists a sequence f_n of nonneg simple functions so that $f_n \nearrow f$ pointwise.

Def: Given $(X, \mathcal{M}, \mu)$, $f: X \to [0,+\infty]$ measurable,

$$\int f \, d\mu := \sup \left\{ \int \varphi \, d\mu : 0 \le \varphi \le f, \ \varphi \text{ simple} \right\}$$

Rmk:

(i) If f is simple, this agrees w/ previous defn

(ii) If $c \geq 0$, $\int cf\,d\mu = c\int f\,d\mu$.

(iii) If $f \leq g$, then $\int f\,d\mu \leq \int g\,d\mu$.

Fix a measure space $(X, \mathcal{M}, \mu)$.

Thm (Monotone Convergence): Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable s.t. $f_n \leq f_{n+1}$ $\forall n$, then

$(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$

$$\lim_{n\to\infty} \int f_n\,d\mu = \int \lim_{n\to\infty} f_n\,d\mu.$$

Pf: First, we show "$\leq$". By hypothesis, $f_n \leq \lim_{n\to\infty} f_n$, so

$$\int f_n\,d\mu \leq \int \lim_{n\to\infty} f_n\,d\mu, \quad \forall n \in \mathbb{N}.$$

Sending $n \to \infty$ gives the result.

Recall $\varphi : X \to [0, +\infty)$

Now, we show "$\geq$". Let φ be a simple function s.t. $0 \leq \varphi \leq \lim_{n\to\infty} f_n$.

Wish: for n sufficiently large...

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int \varphi \, d\mu.$$

Fix $a \in (0,1)$. Note that, if $\varphi(x) \neq 0$,

$$a\varphi(x) < \lim_{n\to\infty} f_n(x)$$

Define $E_n := \{x : f_n(x) \geq a\varphi(x)\} \in \mathcal{M}$.

- Since $f_n \leq f_{n+1}$, $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n$.
- Furthermore $\bigcup_{n=1}^{\infty} E_n = X$, since..
  - if $\varphi(x) = 0$, $x \in E_n \ \forall n \in \mathbb{N}$

□ if $\varphi(x) \neq 0$, then $\forall x \in X$
$\exists N$ s.t. $n > N$ ensures $x \in E_n$.

Thus, we have,

A $\mapsto \int_A \varphi \, d\mu$ is a measure

$$\int f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} a\varphi \, d\mu = \underbrace{a \int_{E_n} \varphi \, d\mu}.$$

By cty from below,

$$\lim_{n \to \infty} \int f_n \, d\mu \geq a \int_X \varphi \, d\mu.$$

Sending $a \to 1$ gives

$$\lim_{n \to \infty} \int f_n \, d\mu \geq \int_X \varphi \, d\mu.$$

Thus $\lim_{n\to\infty} \int f_n \, d\mu$ is either $+\infty$ or a real valued upper bound for the set

$$\left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq \lim_{n\to\infty} f_n, \ \varphi \text{ simple} \right\}$$

In either case,

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int \lim_{n\to\infty} f_n \, d\mu. \ \square$$

Thm (Beppo-Levi): Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable functions, then

$$\sum_{n=1}^{\infty} \int f_n \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu.$$

Pf: First, $f, g : X \to [0, +\infty]$ meas.
There exist $\{\varphi_i\}_{i=1}^{\infty}$, $\{\psi_i\}_{i=1}^{\infty}$
nonneg, simple functions s.t.

$$\varphi_i \nearrow f, \quad \psi_i \nearrow g \text{ pointwise.}$$

Then $\varphi_i + \psi_i \nearrow f+g$. Thus

$$\int f+g \, d\mu = \int \lim_{i\to\infty} \varphi_i + \psi_i \, d\mu$$

$$\overset{MCT}{=} \lim_{i\to\infty} \int \varphi_i + \psi_i \, d\mu$$

$$= \lim_{i\to\infty} \int \varphi_i \, d\mu + \int \psi_i \, d\mu$$

$$\overset{MCT}{=} \int f \, d\mu + \int g \, d\mu.$$

By induction, $\forall N \in \mathbb{N}$,

$$\int \sum_{n=1}^{N} f_n \, d\mu = \sum_{n=1}^{N} \int f_n \, d\mu .$$

By MCT,

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n \, d\mu$$

$$= \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n \, d\mu$$

$$= \sum_{n=1}^{\infty} \int f_n \, d\mu \qquad \square$$

Without <u>any</u> monotonicity of the sequence f_n, you can still get an inequality for the liminfs.

Thm (Fatou's Lemma): Given $\{f_n\}_{n=1}^{\infty} : X \to [0,+\infty]$ measurable,

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu.$$

Pf: By defn $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \underbrace{\inf_{k\geq n} f_k}_{g_n}$

Note that $g_n \leq g_{n+1}$.

Thus, by MCT,

$$\lim_{n\to\infty} \int g_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu.$$

Note that $g_n \leq f_n \; \forall n \in \mathbb{N}$, so

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \liminf_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu.$$

This gives the result. $\square$

Two important examples of strict inequality:

$(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$

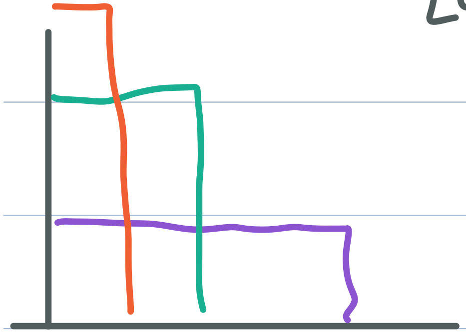
Ex: ("runaway to infinity")

$f_n = 1_{[n, n+1]}$, $\lim_{n\to\infty} f_n = 0$ pointwise

$1 = \liminf_{n\to\infty} \int f_n \, d\lambda > \int \liminf_{n\to\infty} f_n \, d\lambda = 0.$

Ex: ("goes up the spout")

$f_n = n \, 1_{[0, 1/n]}$, $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}$



$$1 = \liminf_{n\to\infty} \int f_n \, d\lambda > \int \liminf_{n\to\infty} f_n \, d\lambda = 0$$

where we use the following proposition ...

Prop: Given $f: X \to [0, +\infty]$ meas,

$$\int f \, d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}$$

$$\Updownarrow$$

$$\mu(\{x : f(x) \neq 0\}) = 0$$

Pf: First, suppose f is simple, i.e.

$$f = \sum_{i=1}^{n} a_i 1_{E_i}, \quad a_i \in \mathbb{R}$$

Then,

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i) = 0$$

$\iff \forall i$, either $a_i = 0$ or $\mu(E_i) = 0$

$\Leftrightarrow f=0$ μ-a.e.

Now, suppose $f: X \to [0,+\infty]$ is an arbitrary meas fn.

$\boxed{\Leftarrow}$

$f=0$ μ-a.e.

$\Rightarrow \varphi = 0$ μ-a.e.

By defn,

$\int f d\mu := \sup\{\int \varphi d\mu : 0 \le \varphi \le f, \varphi \text{ simple}\}$

$=0$

$= 0.$

$\boxed{\Rightarrow}$

Assume that $\mu(\{x : f(x) > 0\}) > 0$.

Note that $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \frac{1}{n}\}$.

By countable subadditivity,

$$0 < \mu(\{x : f(x) > 0\}) \leq \sum_{n=1}^{\infty} \mu(\{x : f(x) > \tfrac{1}{n}\})$$

Thus $\exists\, n \in \mathbb{N}$ s.t. $\mu(\{x : f(x) > \frac{1}{n}\}) > 0$.

Let $\phi = \frac{1}{n} \mathbb{1}_{\{x : f(x) > \frac{1}{n}\}}$.

Then $\phi \leq f$.

Thus,

$$\int f \, d\mu \geq \int \phi \, d\mu = \frac{1}{n} \underbrace{\mu(\{x : f(x) > \tfrac{1}{n}\})}_{> 0}$$

Therefore $\int f \, d\mu > 0$.

This proves the contrapositive. $\square$

Integration of Real-Valued Functions

Measure space $(X, \mathcal{A}, \mu)$.

Given $f: X \to \overline{\mathbb{R}}$, define

"positive part"

$$\left.\begin{aligned} f_+ &= f \vee 0 \\ f_- &= (-f) \vee 0 \end{aligned}\right\} \begin{aligned} f &= f_+ - f_- \\ |f| &= f_+ + f_- \end{aligned}$$

"negative part"

Def: Given $f: X \to \overline{\mathbb{R}}$ meas, if either $\int f_+ \, d\mu$ or $\int f_- \, d\mu$ is finite,

$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu$$

$\int f_+ + f_- \, d\mu < +\infty \iff \int |f| \, d\mu < +\infty$

If both $\int f_+ \, d\mu$ and $\int f_- \, d\mu$ are finite, we say f is integrable and write $f \in L^1(\mu)$.

Prop: $L^1(\mu)$ is a real vector space and

$$f \mapsto \int f \, d\mu$$

is a linear functional on $L^1(\mu)$.

Pf: Next time :)