Math 201a: Midterm 2

Thursday, October 24, 2024

Name: ______________________________

Student ID #: ______________________________

Signature: ______________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	10	
2	10	
3	10	
Total	30	

Question 1 (10 points)

Consider a measure space $(X, \mathcal{M}, \mu)$ and $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$. Recall that

$$\limsup_{i \to +\infty} E_i := \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i \ , \qquad \liminf_{i \to +\infty} E_i := \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} E_i \ .$$

(a) Prove that $\liminf_{i \to +\infty} \mu(E_i) \geq \mu(\liminf_{i \to +\infty} E_i)$.

(b) If $\mu(\cup_{i=1}^{\infty} E_i) < +\infty$, prove that $\limsup_{i \to +\infty} \mu(E_i) \leq \mu(\limsup_{i \to +\infty} E_i)$.

(c) Give a counterexample to show the result from part (b) does not hold without the hypothesis $\mu(\cup_{i=1}^{\infty} E_i) < +\infty$.

Question 2 (10 points)

Suppose $0 < \lambda^*(A) < +\infty$. Prove that, for all $\epsilon > 0$, there exists a nonempty interval $(a, b]$, $a, b \in \mathbb{R}$ so that

$$\lambda^*(A \cap (a, b]) > (1 - \epsilon)\lambda^*((a, b]).$$

Hint: estimate $s := \sup_{a<b\in\mathbb{R}} \lambda^*(A \cap (a, b])/\lambda^*((a, b])$*—for example, can you show that for any covering* $\{(a_i, b_i]\}_{i=1}^{\infty}$ *of* A*, we have* $\lambda^*(A) \leq s \sum_{i=1}^{\infty} \lambda^*((a_i, b_i])$*?*

Question 3 (10 points)

Consider a metric space (X, d) endowed with a σ-algebra $\mathcal{M}$. Parts (a) and (b) are unrelated.

(a) Prove that any lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

(b) Suppose $X = \mathbb{R}$. Prove that any monotone function $f : R \to \overline{\mathbb{R}}$ is Borel measureable.