Midterm 1 Solutions

① ⓐ

By definition of liminf,

$$\liminf_{i\to\infty}\mu(E_i) = \lim_{k\to\infty}\inf_{i\geq k}\mu(E_i)$$

$$\geq \lim_{k\to\infty}\mu\left(\bigcap_{i=k}^{\infty}E_i\right)$$

$$= \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{i=k}^{\infty}E_i\right).$$

monotonicity

cty from below

ⓑ Similarly as in part ⓐ,

$$\limsup_{i\to\infty}\mu(E_i) = \lim_{k\to\infty}\sup_{i\geq k}\mu(E_i)$$

$$\leq \lim_{k\to\infty}\mu\left(\bigcup_{i=k}^{\infty}E_i\right)$$

$$= \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}E_i\right)$$

monotonicity

cty from above

ⓒ Let $X = \mathbb{N}$, $\mathcal{M} = 2^{\mathbb{N}}$, $\mu(E) = |E|$.

Let $E_i = \{n \in \mathbb{N} : n \geq i\}$

then $\limsup_{i\to\infty} E_i = \bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}E_i = \bigcap_{k=1}^{\infty}E_k = \emptyset.$

OTOH, $\limsup_{i\to+\infty}\mu(E_i) = +\infty$. Since

$$\mu(\emptyset) < \limsup_{i\to+\infty}\mu(E_i),$$

this shows that the hypothesis $\mu\left(\bigcup_{i=1}^{\infty}E_i\right) < +\infty$ is necessary in part ⓑ.

(2) We seek to show that

$$s := \sup_{\substack{a,b\in\mathbb{R}\\ a<b}} \frac{\lambda^*(A\cap(a,b])}{\lambda^*((a,b])} \geq 1.$$

If we can show this, then $\forall \varepsilon>0$, $\exists\, a,b\in\mathbb{R}$, $a<b$ s.t.

$$\frac{\lambda^*(A\cap(a,b])}{\lambda^*((a,b])} > 1-\varepsilon,$$

which gives the desired result.

By subadditivity of Lebesgue outer measure, for any $\{(a_i,b_i]\}_{i=1}^{\infty}$ s.t. $A\subseteq \bigcup_{i=1}^{\infty}(a_i,b_i]$,

$$\lambda^*(A) \leq \sum_{i=1}^{\infty} \lambda^*(A\cap(a_i,b_i])$$

$$\leq \sum_{i=1}^{\infty} s\,\lambda^*((a_i,b_i])$$

Thus, by defn of Lebesgue outer measure,

$$\lambda^*(A) \leq s\,\lambda^*(A).$$

Since $\lambda^*(A)<+\infty$, this implies $s\geq 1$.

③

ⓐ

First, suppose f is nondecreasing

It suffices to show that
$f^{-1}((a,+\infty])$ is measurable for all $a \in \mathbb{R}$,
since such sets generate $\mathcal{B}_{\overline{\mathbb{R}}}$.

For all $a \in \mathbb{R}$, $f^{-1}((a,+\infty])$ is a
possibly empty interval in $\mathbb{R}$, since if $x \in f^{-1}((a,+\infty])$,
then $f(x) > a$ and $f(y) > a$ for all $y > x$.

Since all intervals (including the empty set) belong to $\mathcal{B}_{\overline{\mathbb{R}}}$, this gives the result for f nondecreasing.

Now, suppose f is nonincreasing. Then $f = (-1)(-f)$. Since (-1) and $(-f)$ are nondecreasing, they are measurable. Finally, the product of measurable fns is measurable.

(b)

By our defn of a lower semicontinuous function, $f^{-1}((a,+\infty])$ is open, hence Borel measurable, for all $a \in \mathbb{R}$.