Math 201a: Midterm 2

Thursday, November 21, 2024

Name: ______________________________

Student ID #: ______________________________

Signature: ______________________________

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	11	
2	12	
3	7	
Total	30	

Question 1 (11 points)

Suppose $(X, \mathcal{M}, \mu)$ is a measure space and f_n, f are nonnegative $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$-measureable functions satisfying $f_n(x) \to f(x)$ for all $x \in X$. Consider $S \in \mathcal{M}$ arbitrary.

(a) Prove that

$$\int 1_S f d\mu \leq \liminf_{n\to+\infty} \int 1_S f_n d\mu \quad \text{and} \quad \int 1_{S^c} f d\mu \leq \liminf_{n\to+\infty} \int 1_{S^c} f_n d\mu.$$

(b) Use part (a) to show that,

$$\lim_{n\to+\infty} \int f_n d\mu = \int f d\mu < +\infty \implies \lim_{n\to+\infty} \int_S f_n d\mu = \int_S f d\mu, \ \forall S \in \mathcal{M}.$$

Question 2 (12 points)

Compute the following limit and justify your calculations. You do not need to explain why the functions are measurable. Explain where you use the equivalence of the Riemann and Lebesgue integral, as well as where you use standard Calculus facts about the Riemann integral.

$$\lim_{n \to +\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n) d\lambda(x).$$

Question 3 (7 points)

Let X be a nonempty set.

(a) Let $\mathcal{F}$ be a nonempty collection of monotone classes on X. Define

$$\cap\mathcal{F} = \{E : E \in \mathcal{C},\ \forall\mathcal{C} \in \mathcal{F}\}.$$

If $\cap\mathcal{F}$ is nonempty, prove that it is a monotone class.

(b) Given any $\mathcal{E} \subseteq 2^X$, prove that there is a smallest monotone class containing $\mathcal{E}$.