Math 201a: Midterm 2 Practice Problems

(Not to be turned in.)

Question 1

The goal of this question is to prove the following special case of *Lusin's theorem.*

THEOREM 1 (Lusin's Theorem)**.** *Consider a closed interval* $[a, b] \subseteq \mathbb{R}$*. Suppose* $f : [a, b] \to \mathbb{R}$ *is Lebesgue measurable and bounded. For all* $\epsilon > 0$*, there exists a compact set* $E \subseteq [a, b]$ *so that* $\lambda([a, b] \setminus E) < \epsilon$ *and* $f|_E$ *is continuous.*

(a) For any bounded, Lebesgue measurable function $f : [a, b] \to \mathbb{R}$, prove that there exists $\{g_k\}_{k=1}^{\infty} \subseteq C_c(\mathbb{R})$ so that $g_k \to f$ pointwise λ-a.e on $[a, b]$.

(b) For the sequence from part (b), prove that, for all $\epsilon > 0$, there exists a compact set $E \subseteq [a, b]$ so that $\lambda([a, b] \setminus E) < \epsilon$ and $g_k \to f$ uniformly on E.

(c) Apply the results of parts (a) and (b) to complete the proof of the theorem.

Hint: In order to apply Egoroff's theorem above, you will want to consider the restriction of Lebesgue measure to a set E*, that is* $\lambda_E(A) := \lambda(A \cap E)$*.*

Question 2

Consider a measure space $(X, \mathcal{M}, \mu)$ and a sequence of nonnegative, measurable functions f_n satifying

$$f_n(x) \searrow f(x) \text{ for all } x \in X \quad \text{and} \quad \int f_1 < +\infty.$$

Prove that

$$\lim_{n \to +\infty} \int f_n = \int \lim_{n \to +\infty} f_n.$$

Question 3

Let $(X_i, \mathcal{M}_i)_{i=1}^3$ be measurable spaces. Prove that $\bigotimes_{i=1}^3 \mathcal{M}_i = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$.