Math 201a: Practice Midterm 1

(not to be turned in)

Question 1

Fix a measurable space $(X, \mathcal{M})$. Given $f, g : X \to \mathbb{R}$ measurable, prove the following:

(a) $f + g$ is measurable

(b) f^2 is measurable

(c) fg is measurable (Hint: how can you express this in terms of squares and reduce to the previous parts?)

Question 2

Suppose μ and ν are finite measures defined on the same measurable space $(X, \mathcal{M})$. Prove that there exists a set $N \in \mathcal{M}$ with the following properties:

(i) $\mu(N) = 0$;

(ii) if $S \in \mathcal{M}$, $S \subseteq X \setminus N$, and $\mu(S) = 0$, then $\nu(S) = 0$.

Hint: among all sets $N \in \mathcal{M}$ with $\mu(N) = 0$, choose the one for which $\nu(N)$ is largest.

Question 3

In HW2, Q12, you showed that, given an outer measure μ^*, the collection of μ^*-measurable sets $\mathcal{M}_{\mu^*}$ is not necessarily the largest σ-algebra on which μ^* can be restricted to be a measure. In this problem, you will show that, as long as the outer measure of any subset can be approximated by a μ^*-measurable set containing it, then the collection of μ^* measurable sets *is* maximal.

Let X be a nonempty set and suppose μ^* is an outer measure on X. Suppose that, for all $S \subseteq X$ and for all $\epsilon > 0$, there exists a μ^*-measurable set $E \supseteq S$ so that $\mu^*(E) \leq \mu^*(S) + \epsilon$.

(a) Suppose A is not μ^*-measurable and consider the σ-algebra $\mathcal{F}$ generated by $\mathcal{M}_{\mu^*}$ and $\{A\}$. Prove that μ^* is not additive on $\mathcal{F}$.

(b) Use part (a) to conclude that $\mathcal{M}_{\mu^*}$ is the largest σ-algebra on which μ^* can be restricted to be a measure. *(Hint: this is almost immediate from part (a).)*