

Question 2

Homework 1

②

① Suppose $\{x_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathbb{R}}$ converges to $x_* \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_* - \varepsilon, x_* + \varepsilon)$ is an open set containing x_* , $\exists N_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\varepsilon, x_n \in (x_* - \varepsilon, x_* + \varepsilon) \subseteq \mathbb{R}$. Thus, up to modifying the first $N-1$ elements of $\{x_n\}_{n \in \mathbb{N}}$, we have $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$. Furthermore, as above, $x_n \rightarrow x_*$ with respect to the usual topology on $\mathbb{R}$.

Question 2 — continued

Homework 1

Question 2(b)

Proof. Set

$$a_N = \inf_{n \geq N} x_n, \quad b_N = \sup_{n \geq N} x_n.$$

Then, as recalled in Q1,

$$\liminf_{n \rightarrow \infty} x_n = \sup_N a_N, \quad \limsup_{n \rightarrow \infty} x_n = \inf_N b_N.$$

By the definition of the order topology, $x_n \rightarrow L$ in $\overline{\mathbb{R}}$ if and only if both of the following hold: for every real $c < L$, we have $x_n > c$ for n sufficiently large; and for every real $d > L$, we have $x_n < d$ for n sufficiently large. At $L = +\infty$, only the lower-bound condition applies; at $L = -\infty$, only the upper-bound condition applies.

Suppose first that $x_n \rightarrow L$. If $L > -\infty$, for every real $c < L$, we have $x_n > c$ for n sufficiently large, so $a_N \geq c$ for some N . Thus $\liminf_{n \rightarrow \infty} x_n \geq c$ for every real $c < L$, which implies $\liminf_{n \rightarrow \infty} x_n \geq L$. OTOH, if $L = -\infty$, we automatically have $\liminf_{n \rightarrow \infty} x_n \geq L$. Similarly, $\limsup_{n \rightarrow \infty} x_n \leq L$.

By Q1(a), we obtain

$$L \leq \liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n \leq L,$$

so $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = L$.

Conversely, suppose $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n = L$. For any real $c < L$, the identity

$$\sup_N a_N = L > c$$

implies that $a_N > c$ for some N . Hence $x_n \geq a_N > c$ for all $n \geq N$. Likewise, for any real $d > L$, the identity $\inf_N b_N = L < d$ implies that $b_M < d$ for some M , so $x_n \leq b_M < d$ for all $n \geq M$. These are precisely the conditions for $x_n \rightarrow L$. $\square$

Question 2 — continued

Homework 1

Question 2(c)

Proof. Recall that

$$\varphi(x) = \frac{x}{1 + |x|} \quad (x \in \mathbb{R}), \quad \varphi(\pm\infty) = \pm 1.$$

Define $\psi : [-1, 1] \rightarrow \overline{\mathbb{R}}$ by

$$\psi(t) = \frac{t}{1 - |t|} \quad (|t| < 1), \quad \psi(\pm 1) = \pm\infty.$$

Direct substitution shows that $\psi(\varphi(x)) = x$ for every $x \in \overline{\mathbb{R}}$ and $\varphi(\psi(t)) = t$ for every $t \in [-1, 1]$. Thus φ is bijective with inverse ψ .

By the permitted continuity facts, φ is continuous at every finite point and ψ is continuous at every point of $(-1, 1)$. Here we use that the topology on $\mathbb{R}$ inherited from $\overline{\mathbb{R}}$ is the usual topology and that $\mathbb{R}$ is open in $\overline{\mathbb{R}}$.

For $x \geq 0$ and $0 \leq t < 1$, we have

$$\varphi(x) = 1 - \frac{1}{1 + x}, \quad \psi(t) = \frac{1}{1 - t} - 1.$$

Consequently,

$$\lim_{x \rightarrow +\infty} \varphi(x) = 1, \quad \lim_{t \rightarrow 1^-} \psi(t) = +\infty.$$

By the defining neighborhoods of $+\infty$ and 1, these limits establish continuity of φ at $+\infty$ and of ψ at 1, respectively. Since both maps are odd, we also have

$$\lim_{x \rightarrow -\infty} \varphi(x) = -1, \quad \lim_{t \rightarrow -1^+} \psi(t) = -\infty,$$

which establish continuity at the negative endpoints. Hence φ and its inverse are continuous, so φ is a homeomorphism. $\square$

Question 3

Homework 1

③(a) Suppose f is lsc, so that $\{x: f(x) \leq a\}$ is closed $\forall a \in \bar{\mathbb{R}}$. Fix $x_0 \in X$ and $x_n \rightarrow x_0$.

Case 1: $\liminf_{k \rightarrow \infty} f(x_k) = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f(x_n) \neq -\infty$.

For all $\varepsilon > 0$, $\liminf_{k \rightarrow \infty} f(x_k) < \liminf_{k \rightarrow \infty} f(x_k) + \varepsilon$,
so we must have $\inf_{n \geq k} f(x_n) < \liminf_{k \rightarrow \infty} f(x_k) + \varepsilon$
for all $k \in \mathbb{N}$. Thus,

$x_n \in \{x: f(x) \leq \liminf_{k \rightarrow \infty} f(x_k) + \varepsilon\}$ for infinitely many n . As the set is closed, this gives $x_0 \in \{x: f(x) \leq \liminf_{k \rightarrow \infty} f(x_k) + \varepsilon\}$ that is $f(x_0) \leq \liminf_{k \rightarrow \infty} f(x_k) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this gives the result.

Case 2: $\liminf_{k \rightarrow \infty} f(x_k) = -\infty$, so $\inf_{n \geq k} f(x_n) = -\infty \forall k \in \mathbb{N}$.

For all $M \in \mathbb{R}$, $x_n \in \{x: f(x) \leq M\}$
for infinitely many n , so $x_0 \in \{x: f(x) \leq M\}$
that is $f(x_0) \leq M$. Sending $M \rightarrow -\infty$, gives
the result.

Question 3 — continued

Homework 1

Conversely, suppose that for all $x_0 \in X$ and $x_n \rightarrow x_0$, $\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$. For arbitrary $a \in \mathbb{R}$, if $\{x: f(x) \leq a\} = \emptyset$, then $\{x: f(x) > a\} = X$ is open. Alternatively, if $\{x: f(x) \leq a\} \neq \emptyset$, let $\{x_n\}_{n=1}^{\infty} \subseteq \{x: f(x) \leq a\}$ be an arbitrary convergent sequence, $x_n \rightarrow x_0$. Since $\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_0)$, $x_0 \in \{x: f(x) \leq a\}$. Thus, $\{x: f(x) \leq a\}$ is closed, so $\{x: f(x) > a\}$ is open.

(b)

Note that f is upper semicontinuous $\Leftrightarrow \{x: f(x) < a\} = \{x: -f(x) > -a\}$ is open for all $a \in \mathbb{R} \Leftrightarrow -f$ is lower semicontinuous.

By part (a), $-f$ is lower semicontinuous,
 $\Leftrightarrow \liminf_{n \rightarrow \infty} -f(x_n) \geq -f(x_0) \Leftrightarrow -\liminf_{n \rightarrow \infty} -f(x_n) \leq f(x_0)$
 $\stackrel{(a)}{\Leftrightarrow} \limsup_{n \rightarrow \infty} f(x_n) \leq f(x_0)$.

(c) Suppose f is continuous. Then, for every convergent sequence x_n , the discussion at the end of (2),
 $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$.

By parts (a) and (b), this shows f is upper and lower semicontinuous.

Question 3 — continued

Homework 1

Suppose f is upper and lower semicontinuous.
If x_n is a convergent sequence with
limit x_0 ,

$$\limsup f(x_n) \stackrel{(b)}{\leq} f(x_0) \stackrel{(a)}{\leq} \liminf f(x_n).$$

By Q2 (b), this shows $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

Since x_n was arbitrary, this shows f is
continuous.

Question 4

Homework 1

④

(a) It suffices to show the upper and lower Riemann integrals are not equal.

Since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, for any partition P and all $i=1, \dots, n$, there exists $q_i \in \mathbb{Q}$ and $p_i \in \mathbb{R} \setminus \mathbb{Q}$ so that $x_{i-1} \leq q_i \leq x_i$ and $x_{i-1} \leq p_i \leq x_i$. Thus, $M_i = 1$ and $m_i = 0$ for all $i=1, \dots, n$.

Consequently, $U(P, f) = 1$ and $L(P, f) = 0$ for all partitions P . Therefore,

$$\int_a^b f(x) dx = 1 \neq 0 = \int_a^b f(x) dx, \text{ which completes the proof.}$$

Question 6

Homework 1

⑥

① Fix $x \in \mathbb{R}$. Then, $\exists n \in \mathbb{Z}$ s.t. $x \in [n, n+1]$.
Thus $x - n \in [0, 1]$. Since $n \in \mathbb{Q}$, this shows
that the equivalence class $[x]$ contains
an element in $[0, 1]$.

② Take $q_1, q_2 \in \mathbb{Q} \cap [-1, 1]$, and assume
that $A + q_1 \cap A + q_2 \neq \emptyset$. Then $\exists x$ s.t.
 $x = a_1 + q_1$ and $x = a_2 + q_2$ for $a_1, a_2 \in A$.
Thus, $a_1 - a_2 = q_2 - q_1$, so $a_1 \sim a_2$.
By construction of A , this implies
 $a_1 = a_2$. Therefore $q_1 = q_2$. This shows
 $A + q_1$ and $A + q_2$ have nontrivial intersection
only if $q_1 = q_2$. Therefore
 $\{A + q \mid q \in \mathbb{Q} \cap [-1, 1]\}$ is a disjoint collection of
sets.

Question 7

Homework 1

⑦

① $\varepsilon \mapsto \inf \{f(y) : d(x, y) < \varepsilon\}$ is decreasing

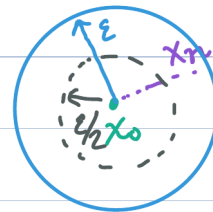
Question 7 — continued

Homework 1

(b) Fix $a \in \mathbb{R}$. We must show $\{x: f(x) \leq a\}$ is closed. Since $\varepsilon \mapsto \inf \{f(y): d(x, y) < \varepsilon\}$ is decreasing, $f(x) \leq a$ iff, for all $\varepsilon > 0$, $\inf \{f(y): d(x, y) < \varepsilon\} \leq a$.

Suppose $\{x_n\}_{n=1}^{\infty} \subseteq \{x: f(x) \leq a\}$ and $x_n \rightarrow x_0$. Then, $\forall \varepsilon > 0, \forall n \in \mathbb{N}, \inf \{f(y): d(x_n, y) < \varepsilon\} \leq a$.

Fix $\tilde{\varepsilon} > 0$ arbitrary. It suffices to show $\inf \{f(y): d(x_0, y) < \tilde{\varepsilon}\} \leq a$.



Choose N so that $d(x_n, x_0) < \frac{\tilde{\varepsilon}}{2} \forall n > N$. Then, $d(x_0, y) < d(x_0, x_n) + d(x_n, y)$, so for $n > N$,

$$\{y: d(x_0, y) < \tilde{\varepsilon}\} \supseteq \{y: d(x_n, y) < \frac{\tilde{\varepsilon}}{2}\}.$$

Thus, for $n > N$,

$$\inf \{f(y): d(x_0, y) < \tilde{\varepsilon}\} \leq \inf \{f(y): d(x_n, y) < \frac{\tilde{\varepsilon}}{2}\} \leq a.$$

This gives the result.

Question 7 — continued

Homework 1

(c) Since $d(x, x) = 0 < \varepsilon$ for all $\varepsilon > 0$, the definition of f_* ensures that $\forall x \in X$

$$f_*(x) = \lim_{\varepsilon \rightarrow 0} \left(\inf \{ f(y) : d(x, y) < \varepsilon \} \right) \leq \lim_{\varepsilon \rightarrow 0} f(x) = f(x).$$

(d) First, note that

$$\inf \left\{ \liminf_{n \rightarrow \infty} f(x_n) : x_n \rightarrow x \right\} \quad \downarrow \text{part (c)}$$

$$\geq \inf \left\{ \liminf_{n \rightarrow \infty} f_*(x_n) : x_n \rightarrow x \right\}$$

$$\geq \inf \left\{ f_*(x) : x_n \rightarrow x \right\} \quad \downarrow \text{part (c)}$$

$$= f_*(x)$$

To see the other inequality, note that by definition of $f_*(x)$, $\forall n \in \mathbb{N}$,

$$f_*(x) \geq \inf \{ f(y) : d(x, y) < \frac{1}{n} \}$$

If $f_*(x) = -\infty$, then the inf is also $-\infty$ for all $n \in \mathbb{N}$, so there exists $x_n \rightarrow x$ so that $f_*(x) = -\infty = \liminf_{n \rightarrow \infty} f(x_n)$. Now assume $f_*(x) > -\infty$.

Next, by definition of the infimum,
 $\forall n \in \mathbb{N}$, $\exists x_n \in X$ s.t. $d(x, x_n) < \frac{1}{n}$ and
 $f_*(x) + \frac{1}{n} \geq f(x_n)$.

By construction, $x_n \rightarrow x$ and

$$f_*(x) \geq \liminf_{n \rightarrow \infty} f(x_n).$$

Question 7 — continued

Homework 1

This shows $f_*(x) \geq \inf \{ \liminf_{n \rightarrow \infty} f(x_n) : x_n \rightarrow x \}$,
which completes the proof.

Question 9

Homework 1

9

$$\begin{aligned} \text{(a)} \quad f^{-1}\left(\bigcup_{\alpha \in A} E_{\alpha}\right) &= \{x \in X : f(x) \in \bigcup_{\alpha \in A} E_{\alpha}\} \\ &= \bigcup_{\alpha \in A} \{x \in X : f(x) \in E_{\alpha}\} \\ &= \bigcup_{\alpha \in A} f^{-1}(E_{\alpha}) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad f^{-1}(E^c) &= \{x \in X : f(x) \in E^c\} \\ &= \{x \in X : f(x) \in E\}^c \\ &= (f^{-1}(E))^c \end{aligned}$$

c) The first is always true, since

$$\begin{aligned} f\left(\bigcup_{\alpha \in A} E_{\alpha}\right) &= \{f(x) : x \in \bigcup_{\alpha \in A} E_{\alpha}\} \\ &= \bigcup_{\alpha \in A} \{f(x) : x \in E_{\alpha}\} \\ &= \bigcup_{\alpha \in A} f(E_{\alpha}). \end{aligned}$$

The second is not always true. Suppose $X=Y=\{0,1\}$ and $f(x)=0 \ \forall x \in X$. Then $f(\{0\}^c) = f(\{1\}) = \{0\} \neq \{1\} = \{0\}^c = f(\{0\})^c$.