

MATH 201A: HOMEWORK 2

Due Sunday, October 11th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice questions will appear on each exam.

Question 1

Prove the following:

- (a) Any algebra that is closed under countable disjoint unions is a σ -algebra.
- (b) Any algebra that is closed under countable increasing unions is a σ -algebra. (We say that an algebra $\mathcal{A}$ is closed under countable increasing unions if, for all $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}$ with $E_i \subseteq E_{i+1}$ for all i , $\cup_{i=1}^\infty E_i \in \mathcal{A}$.)

Question 2

- (a) Prove the following lemma from lecture 2:

LEMMA 1. *Suppose $\mathcal{A}$ is an algebra of subsets of a set X .*

- (i) *If $E_1, \dots, E_n \in \mathcal{A}$, then $\cap_{i=1}^n E_i \in \mathcal{A}$.*
- (ii) *$\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

- (b) Prove the following claim from lecture 2:

Claim: *Given a nonempty collection $\mathcal{C}$ of σ -algebras on X ,*

$$\cap \mathcal{C} := \{E : E \in \mathcal{A}, \forall \mathcal{A} \in \mathcal{C}\}$$

is a σ -algebra.

Question 3

Fix an open set $U \subseteq \mathbb{R}$. Prove that there exist $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subseteq \mathbb{R}$ so that $U = \cup_{n=1}^\infty (a_n, b_n)$.

Question 4*

Prove the following proposition from lecture 2:

PROPOSITION. *The Borel σ -algebra of $\mathbb{R}$ is generated by each of the following:*

- (i) *the half-open intervals: $\mathcal{E}_3 = \{(a, b] : a < b\}$,*
- (ii) *the open rays: $\mathcal{E}_5 = \{(a, +\infty) : a \in \mathbb{R}\}$,*
- (iii) *the closed rays: $\mathcal{E}_7 = \{[a, +\infty) : a \in \mathbb{R}\}$.*

Question 5*

Let X be an uncountable set, and let $\mathcal{A}$ be the collection of subsets $E \subseteq X$ such that either E or E^c is at most countably infinite. Prove that $\mathcal{A}$ is a σ -algebra.

Question 6*

Given a collection of sets $\{E_i\}_{i=1}^\infty$, we may define

$$\limsup_{i \rightarrow +\infty} E_i := \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty E_i ,$$
$$\liminf_{i \rightarrow +\infty} E_i := \bigcup_{k=1}^\infty \bigcap_{i=k}^\infty E_i .$$

We may also consider the sets

$$A_1 := \{x : x \in E_i \text{ for all but finitely many } i\} ,$$
$$A_2 := \{x : x \in E_i \text{ for infinitely many } i\} .$$

Determine for which values of $n, m \in \{1, 2\}$, $\limsup_{i \rightarrow +\infty} E_i = A_n$ and $\liminf_{i \rightarrow +\infty} E_i = A_m$. Justify your answer with a proof.

Question 7

Each $x \in [0, 1]$ has a base-10 decimal expansion $x = \sum_{j=1}^{+\infty} a_j 10^{-j}$, where $a_j = 0, 1, \dots, 9$. This expansion is unique unless x is of the form $p10^{-k}$, for $p, k \in \{0\} \cup \mathbb{N}$, where p is not divisible by 10. In this case x has two expansions: one with $a_j = 0$ for $j > k$ and one with $a_j = 9$ for $j > k$. For the purposes of this problem, whenever two expansions exist for the same number, suppose that we always choose the *standard decimal representation of x* , which is the one for which $a_j = 0$ for $j > k$.

- (a) Fix $n \in \mathbb{N}$. Let A_n be the set of numbers x in the interval $[0, 1]$ with a 7 in the n th decimal place. Prove that A_n belongs to the Borel σ -algebra.
- (b) Let E be the set of numbers x in the interval $[0, 1]$ so that the decimal expansion for x contains a 7. Prove that E belongs to the Borel σ -algebra.
- (c) Let F be the set of numbers x in the interval $[0, 1]$ so that the decimal expansion for x contains infinitely many 7's. Prove that F belongs to the Borel σ -algebra.

Question 8*

Consider a metric space (X, d) and $f : X \rightarrow \overline{\mathbb{R}}$. Recall the notion of lower semicontinuous envelope f_* from Homework 1, Question 7. Let f^* denote the upper semicontinuous envelope of f , which is defined symmetrically. You may use all the analogous properties of the upper semicontinuous envelope f^* that you proved for the lower semicontinuous envelope f_* in Homework 1, Question 7 and Homework 1, Question 8.

- (a) Prove that f is continuous at $x \in X$ if and only if $f_*(x) = f^*(x) = f(x)$.
- (b) Let E be the set of points at which f is discontinuous. Prove that E is a countable union of closed sets, hence a Borel set.

Hint: First prove that $f_*(x) < f^*(x)$ if and only if there exist $p, q \in \mathbb{Q}$ so that $f_*(x) < p < q < f^*(x)$. What does this imply about the set $\{y : f_*(y) < f^*(y)\}$?

Question 9*

Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < +\infty$. Suppose $A_1, A_2, \dots$ are sets in $\mathcal{M}$ with $\mu(A_i) \geq c > 0$ for all i . Let Z be the set of elements $x \in X$ that belong to infinitely many of the A_i 's. Prove that $\mu(Z) \geq c$.

Question 10

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$ nonempty, define the restriction of μ to E by $\mu_E(A) := \mu(A \cap E)$ for all $A \in \mathcal{M}$. Prove that μ_E is a measure on $\mathcal{M}$.

Question 11*

Suppose that X is an uncountable set. Define outer measures μ^* and ν^* on X by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 1 & \text{if } E \text{ is uncountable,} \end{cases}$$
$$\nu^*(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ +\infty & \text{if } E \text{ is uncountable.} \end{cases}$$

- (a) Find all sets that are μ^* measurable. Justify your answer.
- (b) Find all sets that are ν^* measurable. Justify your answer.

Question 12*

By Caratheodory's Theorem, we know that any outer measure μ^* can lead to a measure by restricting μ^* to the collection of μ^* -measurable sets. However, in this problem, we will show that the collection of μ^* -measurable sets is, in general, not the largest σ -algebra on which the restriction of μ^* becomes a measure.

Consider the set $X = \{1, 2, 3\}$. Define an outer measure as follows:

$$\mu^*(A) = \begin{cases} 0 & \text{if } |A| = 0, \\ 1 & \text{if } |A| = 1, 2, \\ 2 & \text{if } |A| = 3. \end{cases}$$

- (a) Prove that μ^* is an outer measure on X .
- (b) Prove that the collection of μ^* measurable sets is $\{\emptyset, X\}$.
- (c) Prove that $\mathcal{A} := \{\emptyset, \{1\}, \{2, 3\}, X\}$ is a σ -algebra.
- (d) Prove that $\mu^*|_{\mathcal{A}}$ is a measure.

This shows that the Caratheodory σ -algebra $\mathcal{M}_{\mu^*}$ is not, in general, the largest σ -algebra on which μ^* can be restricted to be a measure.

Question 13*

In probability, an event E is a subset of the sample space Ω , corresponding to the yes-or-no question “Does the outcome belong to E ?” Logical operations on events correspond to set operations:

$$\begin{aligned} \text{“not } E\text{”} &\longleftrightarrow E^c, \\ \text{“at least one } E_n \text{ occurs”} &\longleftrightarrow \bigcup_{n=1}^{\infty} E_n, \\ \text{“every } E_n \text{ occurs”} &\longleftrightarrow \bigcap_{n=1}^{\infty} E_n. \end{aligned}$$

Given a collection of events $\mathcal{E}$, the generated σ -algebra $\mathcal{M}(\mathcal{E})$ is precisely the smallest collection containing $\mathcal{E}$ and closed under negation (“not”), countable disjunction (“or”), and countable conjunction (“and”). It therefore includes the new questions obtained by repeatedly applying these logical operations to the original events.

Consider two coin tosses, with sample space

$$\Omega = \{HH, HT, TH, TT\},$$

where the first letter records the first toss and the second letter records the second toss. Define

$$\mathcal{F}_1 := \mathcal{M}(\{HH, HT\}), \quad \mathcal{F}_2 := \mathcal{M}(\{HH, TH\}).$$

- List all the events in $\mathcal{F}_1$ and $\mathcal{F}_2$, and determine $\mathcal{F}_1 \cap \mathcal{F}_2$.
- Determine $\mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2)$ and justify your answer.
- Let $D = \{HH, TT\}$ be the event that the two tosses agree. Prove that

$$D \notin \mathcal{F}_1 \cup \mathcal{F}_2, \quad D \in \mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2).$$

Deduce that $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ -algebra.

- Match each description below to the σ -algebra that represents exactly that information. Fill in the blanks using

$$\mathcal{F}_1, \quad \mathcal{F}_2, \quad \mathcal{F}_1 \cap \mathcal{F}_2, \quad \mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2).$$

- _____ contains the questions answerable from the **first toss**.
- _____ contains the questions answerable from the **second toss**.
- _____ contains the questions answerable from **either toss alone**, whichever toss is observed.
- _____ contains the questions answerable from **both observations together**.

This example shows that taking the union of two σ -algebras is not sufficient to answer all questions that are logical consequences of their events—one must take the σ -algebra generated by the union to obtain all logical consequences.