Question 3

Homework 2

(3) Let a_n and b_n be the sequences of all rational numbers so that $(a_n, b_n) \subseteq \mathcal{U}$. Then $\bigcup_{n=1}^{\infty} (a_n, b_n) \subseteq \mathcal{U}$.

To see the opposite containment, fix $u \in \mathcal{U}$. Since $\mathcal{U}$ is open, $\exists\ \varepsilon > 0$ s.t. $(u-\varepsilon, u+\varepsilon) \subseteq \mathcal{U}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a_n, b_n so that $u \in (a_n, b_n) \subseteq (u-\varepsilon, u+\varepsilon)$. Thus $\mathcal{U} \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$.

Question 4

Homework 2

(4)

Recall (from Q3) that we may express any open set $U \subseteq \mathbb{R}$ as $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$ for some $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$. (*)

(i) Since $(a,b] = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n})$, it is clear that $\mathcal{E}_3 \subseteq \mathcal{B}_{\mathbb{R}}$, so $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{B}_{\mathbb{R}}$. OTOH $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}]$, so $(a,b) \subseteq \mathcal{M}(\mathcal{E}_3)$

and (*) ensures $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_3)$.

(ii) Since $(a,b] = (a,+\infty) \setminus (b,+\infty)$, $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_5)$ so part (i) ensures $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$. Since $(a,+\infty) = \bigcup_{n=1}^{\infty} (a, a+n)$, $\mathcal{E}_5 \subseteq \mathcal{B}_{\mathbb{R}}$, so $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{B}_{\mathbb{R}}$.

(iii) Since $[a,+\infty) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, +\infty)$, part (ii) ensures $\mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_5) = \mathcal{B}_{\mathbb{R}}$, so $\mathcal{M}(\mathcal{E}_6) \subseteq \mathcal{B}_{\mathbb{R}}$.

Since $(a,+\infty) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, +\infty) \subseteq \mathcal{M}(\mathcal{E}_7)$, part (ii) ensures $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_6)$.

Question 5

Homework 2

(5) By defn, $\mathcal{A}$ is nonempty.
First, we show closure under complements.
Suppose $E \in \mathcal{A}$. Then either E or E^c is at most countably infinite $\Leftrightarrow$ either E^c or $(E^c)^c$ is at most countably infinite $\Leftrightarrow$ $E^c \in \mathcal{A}$.

Next, we show closure under countable unions. Suppose $E_1, E_2, \ldots \in \mathcal{A}$. Consider $\bigcup_{n=1}^{\infty} E_n$. If $\bigcup_{n=1}^{\infty} E_n$ is at most countably infinite, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. On the other hand, suppose $\bigcup_{n=1}^{\infty} E_n$ is <u>not</u> at most countably infinite. Then there exists at least one set E_m in the sequence that is not countable. By defn of $\mathcal{A}$, E_m^c is at most countably infinite. Thus, $\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \subseteq E_m^c$ is at most countably infinite, so $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

Question 6

Homework 2

⑥ First, we show $\limsup E_i = A_2$. Note that

$x \in A_2 \iff x \in E_i$ for infinitely many i

$\iff \forall k \in \mathbb{N}, \exists i \geq k$ s.t. $x \in E_i$

$\iff \forall k \in \mathbb{N}, x \in \bigcup_{i=k}^{\infty} E_i$

$\iff x \in \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i$

Next, we show $\liminf E_i = A_1$. Note that

$x \in A_1 \iff x \in E_i$ for all but finitely many i

$\iff \exists k \in \mathbb{N}$ s.t. $\forall i \geq k \; x \in E_i$

$\iff \exists k \in \mathbb{N}$ s.t. $x \in \bigcap_{i=k}^{\infty} E_i$

$\iff x \in \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i$

Question 8

Homework 2

(8)

Suppose f is continuous at $x \in X$. Then, for all sequences $x_n \to x$, $\lim_{n\to\infty} f(x_n) = f(x)$.

By HW1, Q7(d), $f_*(x) = \inf\{\liminf f(x_n) : x_n \to x\}$
$= \inf\{f(x)\}$
$= f(x)$.

Similarly, $f^*(x) = \sup\{\limsup f(x_n) : x_n \to x\} = f(x)$.

Now, suppose $f_*(x) = f^*(x) = f(x)$. Then

$f_*(x) = \inf\{\liminf f(x_n) : x_n \to x\}$
$\leq \inf\{\limsup f(x_n) : x_n \to x\}$
$\leq \sup\{\limsup f(x_n) : x_n \to x\}$
$= f^*(x)$
$= f_*(x)$

Thus, equality holds throughout and, $\forall x_n \to x$, $\limsup f(x_n) = f^*(x)$. Similarly, $\forall x_n \to x$, $\liminf f(x_n) = f(x)$. Therefore, by HW1, Q2(b), $\lim_{n\to\infty} f(x_n) = f(x)$. This shows f is continuous at x.

(b)

$$E = \{x : f_*(x) < f^*(x)\}$$

$$= \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} \{x : f_*(x) \le p\} \cap \{x : q \le f^*(x)\}$$

since f_* is lsc, this is closed; since f^* is usc, this is closed

$$= \bigcup_{\substack{p,q \in \mathbb{Q} \\ p<q}} f_*^{-1}((-\infty, p]) \cap (f^*)^{-1}([q, +\infty))$$

This is a countable union of closed sets.

Question 9

Homework 2

(9)

By Q6,

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$$

Define $B_n = \bigcup_{k \geq n} A_k$. Then $B_1 \supseteq B_2 \supseteq \dots$

and $\mu(B_1) \leq \mu(X) < +\infty$. Thus,

$$\mu(Z) = \mu\left(\bigcap_{n=1}^{\infty} B_n\right) \overset{\text{cty from above}}{=} \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu\left(\bigcup_{k\geq n} A_k\right)$$

$$\geq \lim_{n\to\infty} \mu(A_n) \geq c.$$

Question 11

Homework 2

(11)

(a)

Claim: $\mathcal{M} = \{A \subseteq X : \text{either } A \text{ or } A^c \text{ is countable}\}$ is the collection of μ^*-measurable sets.

Assume $A \in \mathcal{M}$. If A is countable, $\mu^*(A) = 0$, hence A is μ^*-measurable. If A^c is countable, then the preceding sentence shows A^c is μ^*-measurable, and since the collection of μ^*-measurable sets is a σ-algebra, A must also be μ^*-measurable.

Now, suppose $A \notin \mathcal{M}$. Then neither A nor A^c is countable. Consequently, $\mu^*(X) = 1 \neq 2 = \mu^*(X \cap A) + \mu^*(X \cap A^c)$. Thus A is not μ^*-measurable.

(b)

Claim $\mathcal{M} := 2^X$ is the collection of ν^*-measurable sets

Fix $A \subseteq X$.

If E is countable, so are $E \cap A$ and $E \cap A^c$, hence $\nu^*(E) = 0 = \nu^*(E \cap A) + \nu^*(E \cap A^c)$.

If E is uncountable, then either $E \cap A$ or $E \cap A^c$ is uncountable. Thus, $\nu^*(E) = +\infty = \nu^*(E \cap A) + \nu^*(E \cap A^c)$.

(12)

(a) By definition $\mu^*(\emptyset)=0$ and $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$. Given $\{A_i\}_{i=1}^{\infty} \subseteq 2^X$, first, suppose $|\bigcup_{i=1}^{\infty} A_i| = 0$. Then $|A_i|=0 \ \forall i$, so

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Next, suppose $|\bigcup_{i=1}^{\infty} A_i| = 1$.
Then there must be some i s.t. $|A_i|=1$.
Thus

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Next, suppose $|\bigcup_{i=1}^{\infty} A_i| = 2$.
Then there must either be $i \neq j$ s.t. $|A_i|=|A_j|=1$ or i s.t. $|A_i|=2$. In both cases,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

Lastly, suppose $|\bigcup_{i=1}^{\infty} A_i| = 3$.
Then there must either be i,j,k distinct with $|A_i|=1$ or i and j distinct with $|A_i|=2$, $|A_j|=1$ or k s.t. $|A_k|=3$. In all cases,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 2 \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

(b) Suppose $A \subseteq X$ has one element. WLOG, $A = \{1\}$. Then,

$$\mu^*(\{1,2\}) = 1 \neq 1+1 = \mu^*(\{1,2\}\cap A) + \mu^*(\{1,2\}\cap A^c\}$$

Thus A and A^c are not μ^*-meas. This shows $\mathcal{M}_{\mu^*} = \{\emptyset, X\}$.

(c) $\mathcal{A}$ is closed under complements and arbitrary unions. Thus, it is a σ-algebra.

(d) It suffices to show $\mu^*|_{\mathcal{A}}$ is countably additive. The only nontrivial disjoint unions are of the form $\{1\}\cup\{2,3\}$. In this case,

$$\mu^*|_{\mathcal{A}}(\{1\}\cup\{2,3\}) = 2 = 1+1 = \mu^*|_{\mathcal{A}}(\{1\}) + \mu^*|_{\mathcal{A}}(\{2,3\}).$$

Question 13

Homework 2

Write $A = \{HH, HT\}$ and $B = \{HH, TH\}$.

(a) A σ-algebra containing A must contain $\varnothing$, Ω, and A^c. Conversely, $\{\varnothing, \Omega, A, A^c\}$ is a σ-algebra. Applying the same argument to B gives

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{HH, HT\}, \{TH, TT\}\},$$
$$\mathcal{F}_2 = \{\varnothing, \Omega, \{HH, TH\}, \{HT, TT\}\}.$$

The only events common to these two collections are $\varnothing$ and Ω. Thus, $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\varnothing, \Omega\}$.

(b) Let $\mathcal{G} = \mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2)$. Since $A, B \in \mathcal{G}$, closure under complements and finite intersections implies that

$$\{HH\} = A \cap B, \qquad \{HT\} = A \cap B^c,$$
$$\{TH\} = A^c \cap B, \qquad \{TT\} = A^c \cap B^c$$

all belong to $\mathcal{G}$. Every subset of Ω is a finite union of these singletons, so $\mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2) = 2^{\Omega}$.

(c) The event $D = \{HH, TT\}$ satisfies $D \notin \mathcal{F}_1 \cup \mathcal{F}_2$. On the other hand,

$$D = (A \cap B) \cup (A^c \cap B^c) \in \mathcal{G}.$$

If $\mathcal{F}_1 \cup \mathcal{F}_2$ were a σ-algebra, its closure under these operations would force it to contain D, so $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.

(d) $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{M}(\mathcal{F}_1 \cup \mathcal{F}_2)$.