Question 2

Homework 3

(2)

(a) First, suppose $\mu^*(A) = +\infty$. Then we must have $\mu(B) = +\infty$ for all B s.t. $A \subseteq B$, $B \in \mathcal{A}$. In particular, $B = X$ attains the infimum.

Now, suppose $\mu^*(A) < +\infty$.
Take a minimizing sequence, that is a sequence B_n s.t. $A \subseteq B_n$, $B_n \in \mathcal{A}$ and $\lim_{n\to\infty} \mu(B_n) = \mu^*(A)$. WLOG, we may assume $\mu(B_n) < +\infty \ \forall n \in \mathbb{N}$.

Define $C = \bigcap_{n=1}^{\infty} B_n$. Then $C \subseteq B_n \ \forall n$, $C \in \mathcal{A}$ and $A \subseteq C$. Thus,

$$\mu^*(A) = \lim_{n\to\infty} \mu(B_n) \geq \mu(C) \geq \mu^*(A) \qquad (C \subseteq B_n)$$

Thus, equality must hold throughout, so C attains the minimum.

(b) To see $\mu^*(\emptyset)=0$, take $B=\emptyset$ in the definition of μ^*.

If $E \subseteq F$, then the **constraint** set for F (i.e. the collection of sets $\{B_i\}_{i=1}^{+\infty} \subseteq \mathcal{A}$ s.t. $F \subseteq \bigcup_i B_i$) is contained in the **constraint** set for E. Thus, the inf over the larger set is smaller: $\mu^*(E) \leq \mu^*(F)$.

Fix $\{A_i\}_{i=1}^{+\infty} \subseteq 2^X$. Choose $\{B_i\}_{i=1}^{+\infty} \subseteq \mathcal{A}$ that attain the minimum in the defn of $\mu^*(A_i)$.

Then, since $\bigcup_{i=1}^{+\infty} A_i \subseteq \bigcup_{i=1}^{+\infty} B_i$, $\bigcup_{i=1}^{+\infty} B_i \in \mathcal{A}$,

$$\mu^*\left(\bigcup_{i=1}^{+\infty} A_i\right) \leq \mu\left(\bigcup_{i=1}^{+\infty} B_i\right) \leq \sum_{i=1}^{+\infty} \mu(B_i) = \sum_{i=1}^{+\infty} \mu^*(A_i).$$

(countable subadditivity of μ)

This shows μ^* is an outer measure.

(c) Suppose $A \in \mathcal{A}$. Fix arbitrary $E \subseteq X$. Choose F that attains the minimum in the defn of $\mu^*(E)$. Then,

(countable additivity of μ)

$$\mu^*(E) = \mu(F) = \mu(F \cap A) + \mu(F \cap A^c) \qquad (*)$$

($F \cap A \in \mathcal{A}$, $E \cap A \subseteq F \cap A$; $F \cap A^c \in \mathcal{A}$, $E \cap A^c \subseteq F \cap A^c$)

$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

so A is μ^*-measurable. Choosing $E = A$, in equation (✱), we obtain

$$\mu^*(A) = \underbrace{\mu(F \cap A)}_{= \mu(A), \text{ since } E \subseteq F} + \overbrace{\mu(F \cap A^c)}^{\geq 0} \geq \mu(A).$$

The opposite inequality is true by defn of μ^*.

choose $B_1 = B$, $B_n = \emptyset \ \forall n > 1$

(d) By definition, $\tilde{\mu}^*(A) \leq \mu^*(A)$, so it suffices to show the opposite inequality.

Fix $A \subseteq X$. Choose $\{B_i\}_{i=1}^{+\infty} \subseteq \mathcal{A}$ s.t. $A \subseteq \bigcup_{i=1}^{+\infty} B_i$. Then,

$$\sum_{i=1}^{+\infty} \mu(B_i) \geq \mu\left(\bigcup_{i=1}^{+\infty} B_i\right) \geq \mu^*(A).$$

countable subadditivity

Taking the infimum over the left hand side gives the result.

Question 3

Homework 3

(3)

Define $M := \sup\{\mu^*(B) : B \subseteq A, B \ \mu^*\text{-meas}\}$. (*)
Since $\mu^*(A) < +\infty$, monotonicity ensures $M < +\infty$.

Furthermore, since $\mu^*(B) \geq 0 \ \forall B$, we have $M \geq 0$.

Let B_n be a maximizing sequence, that is $B_n \subseteq A$, B_n μ^*-meas $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu^*(B_n) = M$.

Define $B = \bigcup_{n=1}^{\infty} B_n$. Then B is μ^*-measurable $B \subseteq A$, and $B_n \subseteq B$. Thus
$$M \geq \mu^*(B) \geq \mu^*(B_n).$$

Taking limits gives,

$$M \geq \mu^*(B) \geq \lim_{n\to\infty} \mu^*(B_n) = M.$$

Thus, B attains the maximum in (*).

Consider the set $S := A \setminus B$. Since B is μ^*-measurable and $A = S \cup B$, S must not be μ^*-measurable, or else A would be, too. Thus, $\mu^*(S) > 0$.

Suppose $T \subsetneq S$ and T is μ^*-measurable. Then $B \cup T \subseteq A$ and $B \cup T$ is μ^*-measurable, so by definition of M,

$$M \geq \mu^*(B \cup T).$$

On the other hand, by the μ^*-measurability of B,

$$M \geq \mu^*(B \cup T) = \mu^*((B \cup T) \cap B) + \mu^*((B \cup T) \cap B^c)$$

$\downarrow T \subsetneq S = A \setminus B$

$$= \mu^*(B) + \mu^*(T)$$

$$= M + \mu^*(T).$$

Thus, $\mu^*(T) = 0$.

Question 8

Homework 3

(8)

CLAIM: If $p, q \in \mathbb{Q}$, $p \neq q$, then $A+p \cap A+q = \emptyset$

Pf of CLAIM: Suppose $\exists\, x \in A+p \cap A+q$. Then $x = a_1 - p = a_2 - q$ for $a_1, a_2 \in A$, $a_1 \neq a_2$. This contradicts the definition of A as a set with the property that for every $x \in \mathbb{R}$, there is exactly one $y \in A$ s.t. $x - y \in \mathbb{Q}$.

(a)

By CLAIM, $\bigcup_{q \in \mathbb{Q}} A+q$ is a disjoint union and

$$(0,1] \subseteq \bigcup_{q \in \mathbb{Q}} A+q$$

By countable subadditivity, monotonicity, and translation invariance of Lebesgue outer measure,

$$1 = \lambda^*((0,1]) \leq \lambda^*\left(\bigcup_{q \in \mathbb{Q}} A+q\right) = \sum_{q \in \mathbb{Q}} \lambda^*(A+q) = \sum_{q \in \mathbb{Q}} \lambda^*(A)$$

Thus, $\lambda^*(A) > 0$.

(b)

Suppose $S \in \mathcal{M}_{\lambda^*}$ and $S \subseteq A$. Then the CLAIM ensures $\bigcup_{n=1}^{\infty} S + \frac{1}{n}$ is a disjoint union.

By countable additivity, monotonicity, and translation invariance of Lebesgue measure,

$$\sum_{n=1}^{\infty} \lambda(S) = \sum_{n=1}^{\infty} \lambda\left(S+\tfrac{1}{n}\right) = \lambda\left(\bigcup_{n=1}^{\infty} S+\tfrac{1}{n}\right) \le \lambda((-1,2]) = 3.$$

$S+\frac{1}{n} \subseteq A+\frac{1}{n} \subseteq [0,1]+\frac{1}{n}$

Thus, $\lambda(S)=0$.

(c) Suppose A were Lebesgue measurable. Then part (b) ensures $\lambda^*(A)=0$. This is a contradiction.

Comment: Since $S \subseteq [0,1]$, every translate $S + 1/n$ lies in $[0,2]$. Thus the upper bound 3 can be replaced by $\lambda([0,2]) = 2$. Either finite bound gives the same conclusion, $\lambda(S) = 0$.

Question 9

Homework 3

(9)

(a) WTS $E \in \mathcal{M}_{\mu^*} \iff \exists\, A \in \mathcal{A}$ s.t. $E \subseteq A$, $\mu^*(A \setminus E) = 0$.

First, we show "$\Leftarrow$". Fix $S \subseteq X$. Then, for all $\varepsilon > 0$, $\exists\, B_\varepsilon \in \mathcal{A}$ s.t. $S \subseteq B_\varepsilon$ and

$$\mu^*(S) \geq \mu(B_\varepsilon) - \varepsilon$$

additivity of μ

$$= \mu(B_\varepsilon \cap A) + \mu(B_\varepsilon \cap A^c) - \varepsilon$$

$$= \mu^*(B_\varepsilon \cap A) + \mu^*(B_\varepsilon \cap A^c) - \varepsilon$$

monotonicity

$$\geq \mu^*(S \cap E) + \mu^*(S \cap A^c) - \varepsilon$$

$$\geq \mu^*(S \cap E) + \mu^*(S \cap E^c) - \varepsilon.$$

where the last inequality follows from the fact that

subadd

$$\mu^*(S \cap E^c) \leq \mu^*(S \cap A^c) + \mu^*(S \cap E^c \cap A)$$

$$\leq \mu^*(S \cap A^c) + \underbrace{\mu^*(A \setminus E)}_{0 \text{ by assumption}}.$$

Sending $\varepsilon \to 0$ in (★) shows $E \in \mathcal{M}_{\mu^*}$.

Now, we show "$\Rightarrow$". Choose B that attains the infimum in the definition of the outer measure (see HW3, Q2(a)), so

$E \subseteq B$, $B \in \mathcal{A}$, $\mu^*(E) = \mu(B)$.

Then,

$$\mu(B) \overset{\text{since } E \in \mathcal{M}_{\mu^*}}{=} \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

$$= \mu^*(E) + \mu^*(B \setminus E).$$

If $\mu^*(E) < +\infty$, this gives the result.

Finally, suppose $\mu^*(E) = +\infty$. Since μ is σ-finite, there exist $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $\bigcup_i E_i = X$ and $\mu(E_i) < +\infty$.

By monotonicity, $\mu^*(E \cap E_i) \leq \mu^*(E_i) = \mu(E_i) < \infty$, and since $\mathcal{A} \subseteq \mathfrak{M}_{\mu^*}$ (HW3, Q2(c)), $E \cap E_i \in \mathfrak{M}_{\mu^*}$. By what we have just shown, $\exists \{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ s.t. $E_i \cap E \subseteq A_i$ and

$$0 = \mu^*(A_i \setminus (E_i \cap E)) = \mu^*(A_i \cap (E_i^c \cup E^c)).$$

Let $A = \bigcup_{i=1}^{\infty} A_i$. Then $E \subseteq A$, $A \in \mathcal{A}$ and

$$\mu^*(A \setminus E) = \mu^*(A \cap E^c) \overset{\text{subadd}}{\leq} \sum_i \mu^*(A_i \cap E^c)$$

$$\overset{\text{monotonicity}}{\leq} \sum_i \mu^*(A_i \cap (E_i^c \cup E^c))$$

$$= 0.$$

□

(b) By part (a), if $E \in \mathfrak{M}_{\mu^*}$, then $E^c \in \mathfrak{M}_{\mu^*}$, so $\exists A \in \mathcal{A}$ s.t. $E^c \subseteq A$ and

$$0 = \mu^*(A \setminus E^c) = \mu^*(A \cap E) = \mu^*(E \setminus A^c).$$

Thus $A^c \subseteq E$ and $E = A^c \cup (E \setminus A^c)$

Furthermore, by HW3, Q2(b), $\exists N \in \mathcal{A}$ s.t. $E \setminus A^c \subseteq N$ and $\mu(N) = \mu^*(E \setminus A^c) = 0$.

Correction: "HW3, Q2(b)" should read "HW3, Q2(a)" (attainment of the infimum).

This shows $E \in \{A \cup B : A \in \mathcal{A} \text{ and } B \subseteq N \text{ for } N \in \mathcal{N}\}$.

Conversely, if $E = A \cup B$, for A and B as above, then defining $F = A \cup N$, we see $E \subseteq F$, $F \in \mathcal{A}$ and

$$\mu^*(F \setminus E) = \mu^*((A \cup N) \cap (A^c \cap B^c))$$

$$= \mu^*(N \cap (A^c \cap B^c))$$

$$\leq \mu^*(N)$$

$$= \mu(N)$$

$$= 0,$$

so $E \in \mathcal{M}_{\mu^*}$ by part (a).

(c) By 62(d), it is clear that

$$\mu_F^*(A) \geq \mu^*(A),$$

since we may take the B_i to be half open intervals. On the other hand, for any $B \supseteq A$, $B \in \mathcal{B}_R$,

$$\mu_F^*(A) \le \mu_F^*(B) = \mu(B).$$

Taking the infimum over all such B gives

$$\mu_F^*(A) \le \mu^*(A).$$

Homework 3, Question 9(c) — continued

We have shown that $\mu^* = \mu_F^*$ for $F(x) = x$. Since $\mu = \lambda$ on $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$ is σ-finite, part (b) gives

$$\overline{\mathcal{A}} = \mathcal{M}_{\mu^*} = \mathcal{M}_{\mu_F^*}.$$

Thus the two measures have the same domain.

Let $E \in \overline{\mathcal{A}}$. By definition, $E = A \cup B$, where $A \in \mathcal{A}$ and $B \subseteq N$ for some $N \in \mathcal{N}$. By monotonicity and Question 2(c),

$$0 \leq \mu^*(B) \leq \mu^*(N) = \mu(N) = 0.$$

Therefore,

$$\mu(A) = \mu^*(A) \leq \mu^*(E) \leq \mu^*(A) + \mu^*(B) = \mu(A).$$

It follows that

$$\bar{\mu}(E) = \mu(A) = \mu^*(E) = \mu_F^*(E).$$

Hence the completion $\bar{\mu}$ of μ coincides with the restriction of Lebesgue outer measure to the Lebesgue measurable sets.