Math 201A: Homework 4 Solutions

Question 2

(a)

We proceed by induction.

Base case: $n=0$, $C_0=[0,1]$ is the union of $2^0=1$ disjoint closed interval of length $(\frac{1}{3})^0=1$.

Inductive step: Assume the result holds for C_{n-1}. By definition C_n is formed from C_{n-1} by removing the open middle third of each interval. Thus the number of intervals doubles and the length of each interval is a third of what it used to be. Also, all of the intervals remain closed and disjoint. Thus, C_n is the union of $2\cdot 2^{n-1}=2^n$ disjoint, closed intervals, each of length $(\frac{1}{3})\cdot(\frac{1}{3})^{n-1}=(\frac{1}{3})^n$.

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Question 2 (continued)

(b) Suppose $x, y \in C$, $x < y$. Choose $n \in \mathbb{N}$ s.t. $\frac{1}{3^n} < |x-y|$. In the set C_n, there are no intervals longer than $\frac{1}{3^n}$. Thus, x and y belong to disjoint intervals in C_n. Hence $\exists z \in (x,y)$ s.t. $z \notin C$.

(c) Each C_n is a finite union of closed intervals, hence is closed. Since C is an intersection of closed sets, it is closed.

(d) By (c), $\text{int}(\text{cl}(C)) = \text{int}(C)$. By part (b), $\text{int}(C) = \emptyset$.

(e) Since C is closed $C \in \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\lambda^*}$, so C is Lebesgue measurable.

By definition $C_1 \supseteq C_2 \supseteq \dots$ and $\lambda(C_1) = \lambda([0,1]) = 1$. Thus, by continuity from above,

$$\lambda(C) = \lambda\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \lambda(C_n)$$

$$\overset{(a)}{=} \lim_{n\to\infty} \sum_{i=1}^{2^n} \left(\frac{1}{3}\right)^n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

Typographical correction in (e): the finiteness check should read $\lambda(C_0) = \lambda([0,1]) = 1$.

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Question 3

(a) Write $D = [0,1]\backslash C$. By definition, this function is non-decreasing and strictly bounded between 0 and 1 on D. Consequently, for $x \in D$, $f(x) = \sup\{f(t) : t \in D,\ t < x\}$: since f is increasing, it is clearly an upper bound for the set; furthermore, no smaller number than $f(x)$ can be an upper bound since $f(x)$ is an element of the set (for every $x \in D$, there is some t in the same interval as x satisfying $t < x$). Thus $f(x) = \sup\{f(t) : t \in D,\ t < x\}$ for all $x \in D$. The sets over which the supremum is taken increase with x, so f is nondecreasing on all of $[0,1]$.

By definition of f on $[0,1] \setminus C$, we see that f takes the value $(2^n - 1)/2^n$ for all n. Since 1 is an upper bound for the values of f on $[0,1] \setminus C$, $f(1) \leq 1$. Since f is increasing, $f(1) \geq 1$. By the hint, if f were discontinuous, its range would omit a nonempty open interval. Such an interval would contain a number of the form $(2k-1)/2^n$ for some $k = 1, \ldots, 2^{n-1}$ and $n \in \mathbb{N}$. This value is attained by f, a contradiction. Hence f is continuous.

(b) The function $g(x) = x + f(x)$ is continuous and strictly increasing, with $g(0) = 0$ and $g(1) = 2$. The intermediate value theorem therefore makes g a bijection from $[0,1]$ onto $[0,2]$.

(c) For $x < y$ we have $g(y) - g(x) \geq y - x$, so

$$|g^{-1}(u) - g^{-1}(v)| \leq |u - v| \qquad (u, v \in [0,2]).$$

Thus g^{-1} is 1-Lipschitz, hence continuous.

(d) The set $g(C)$ is compact, hence Borel. Write $D = \bigsqcup_k I_k$ for the open intervals I_k deleted from C_{n-1} to construct C_n for some $n \in \mathbb{N}$. On each I_k, f is a constant c_k, so $g(I_k) = I_k + c_k$. The images are disjoint because g is injective. Since $\lambda(C) = 0$, countable additivity and translation invariance give

$$\lambda([0,2] \setminus g(C)) = \sum_k \lambda(g(I_k)) = \sum_k \lambda(I_k) = \lambda(D) = 1.$$

Therefore $\lambda(g(C)) = 2 - 1 = 1$.

(e) By Homework 3, Question 5, choose a non-Lebesgue-measurable set $A \subseteq g(C)$. Then $B = g^{-1}(A) \subseteq C$ is Lebesgue measurable by completeness. If B were Borel, continuity of $g^{-1} : [0,2] \to [0,1]$ would make $A = (g^{-1})^{-1}(B)$ Borel, a contradiction. Thus $B \in \mathcal{M}_{\lambda^*} \setminus \mathcal{B}_{\mathbb{R}}$.

(f) The Borel null set C has the non-Borel subset B defined in the solution to (e). Hence Lebesgue measure restricted to the Borel sets is not complete.

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(a) Fix $a \in \mathbb{R}$. It suffices to show

$(f+g)^{-1}((a,+\infty)) \in \mathcal{M}$.

By definition,

$$(f+g)^{-1}((a,+\infty)) = \{x : f(x)+g(x) > a\}$$

$$= \bigcup_{r \in \mathbb{Q}} \{x : f(x) > r > a - g(x)\}$$

$$= \bigcup_{r \in \mathbb{Q}} \{x : f(x) > r\} \cap \{x : g(x) > a - r\}$$

$$= \bigcup_{r \in \mathbb{Q}} f^{-1}((r,+\infty)) \cap g^{-1}((a-r,+\infty))$$

Since f, g are measurable fns, this set belongs to $\mathcal{M}$.

(b) Fix $a \geq 0$.

$$(f^2)^{-1}([a,+\infty)) = \{x : f(x)^2 \geq a\}$$

$$= \{x : f(x) \geq \sqrt{a}\} \cup \{x : f(x) \leq -\sqrt{a}\}$$

By measurability of f, this set belongs to $\mathcal{M}$.

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OTOH, if $a < 0$,

$$(f^2)^{-1}([a,+\infty)) = (f^2)^{-1}([0,+\infty)).$$

Thus, $\forall a \in \mathbb{R}$, $(f^2)^{-1}([a,+\infty)) \in \mathcal{M}$.
Since $\{[a,+\infty) : a \in \mathbb{R}\}$ generate $\mathcal{B}_{\mathbb{R}}$, this shows f^2 is measurable.

(c) $fg = \frac{1}{2}(f+g)^2 - f^2 - g^2$

By part (a), sums of measurable functions are measurable.

By part (b), squares of measurable functions are measurable.

By class, h measurable $\Rightarrow -h$ meas.

Combining these facts, we obtain fg is measurable.

Typographical correction in (c): $fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big)$.

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