Math 201A: Homework 5

Due Monday, November 2, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1

Consider a measurable space $(X, \mathcal{M})$. Given $f, g : X \to \bar{\mathbb{R}}$ measurable, prove the following:

(a) fg is measurable (where $0 \cdot \pm\infty = 0$)

(b) Fix $a \in \bar{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Question 2*

(a) Consider a metric space (X, d). Prove that any lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ is $(\mathcal{B}_X, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$g(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is Borel measurable. (Hint: Recall HW1, Q8(a).)

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let D be the set of points at which f is differentiable. Prove that $D \in \mathcal{B}_{\mathbb{R}}$.

Question 3*

Let μ be a locally finite measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, that is, $\mu(B) < +\infty$ for every ball $B \subseteq \mathbb{R}$. Define

$$B(x, r) = \{y : |y - x| < r\}, \quad \bar{B}(x, r) = \{y : |y - x| \leq r\}.$$

(a) Prove that, for all $r > 0$, $x \mapsto \mu(B(x, r))$ is lower semicontinuous.
Hint: First, prove that $\mu(B(x, r)) = \lim_{n \to +\infty} \mu(B(x, r - 1/n))$.

(b) Prove that, for all $r > 0$, $x \mapsto \mu(\bar{B}(x, r))$ is upper semicontinuous.
Hint: How can you adapt the hint from part (a)?

(c) For any $k \geq 0$, prove that

$$f(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{r^k}$$

is Borel measurable.
Hint: Write $f(x) = \lim_{r \to 0} G_r(x)$, *where*

$$G_r(x) = \sup \left\{ \frac{\mu(B(x, \rho))}{\rho^k} : 0 < \rho < r \right\}.$$

© Katy Craig, 2026

Question 4*

Give an example of an *uncountable* collection of Borel functions such that the pointwise supremum is *not* a Borel function.

Hint: consider the indicator function on a nonmeasurable set A, defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Question 5

(a) Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, $\psi : \mathbb{R} \to \mathbb{R}$, and $\phi = \psi$ Lebesgue almost everywhere. Prove that ψ is Lebesgue measurable.

(b) Now, suppose $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable, $\psi : \mathbb{R} \to \mathbb{R}$, and $\phi = \psi$ Lebesgue almost everywhere. Is ψ Borel measurable? Justify your answer with a proof or counterexample.

Question 6

LEMMA 1. *If μ is a Lebesgue-Stieltjes measure on $\mathbb{R}$ and $E \in \mathcal{M}_{\mu^*}$ with $\mu(E) < +\infty$, then for all $\epsilon > 0$, there exists a finite collection of disjoint open intervals $\{I_i\}_{i=1}^n$ so that*

$$\mu\left(E \Delta \left(\cup_{i=1}^n I_i\right)\right) < \epsilon.$$

Prove the lemma.

Question 7*

Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f : X \to \overline{\mathbb{R}}$ is measurable.

(a) If $f \geq 0$ and $\int f d\mu < +\infty$, prove that $\{x : f(x) = +\infty\}$ is a null set.

(b) Under the same hypotheses of part (a), prove that $\{x : f(x) > 0\}$ is σ-finite.

(c) If $\int |f| \, d\mu < +\infty$, prove that there exists $g : X \to \mathbb{R}$ so that g is measurable and $g = f$ μ-a.e..

Question 8*

Consider a measure space $(X, \mathcal{M}, \mu)$ and a nonnegative, measurable function $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists $A \in \mathcal{M}$ so that $\mu(A) < +\infty$ and $\int_A f d\mu > \int f d\mu - \epsilon$.

Hint: use the definition of the integral in terms of simple functions, and choose a simple function with integral close to $\int f d\mu$.

© Katy Craig, 2026

Question 9

Consider a measure space $(X, \mathcal{M}, \mu)$ and a sequence of nonnegative, measurable functions f_n satisfying

$$f_n(x) \searrow f(x) \text{ for all } x \in X \quad \text{and} \quad \int f_1 < +\infty.$$

Prove that

$$\lim_{n\to+\infty} \int f_n = \int \lim_{n\to+\infty} f_n.$$

Question 10*

Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to [0, +\infty]$ satisfies $\int f \, d\mu < +\infty$. Prove that, for all $\epsilon > 0$, there exists $\delta > 0$ so that if $E \in \mathcal{M}$ satisfies $\mu(E) < \delta$, then $\int_E f d\mu < \epsilon$.

Hint: Proceed by contradiction, so that you can find a sequence $E_n \in \mathcal{M}$ with arbitrarily small measure so that $\int_{E_n} f d\mu \geq \epsilon$. Consider the set $A = \cap_{n=1}^{+\infty} \cup_{k>n} E_k$. Use Question 9.

Question 11

Consider a measure space $(X, \mathcal{M}, \mu)$.

(a) Fix a nonnegative, measurable function f. Prove that $\nu(A) := \int_A f d\mu$ is a measure on $\mathcal{M}$.

(b) Suppose $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$, where λ is Lebesgue measure. Fix $f : \mathbb{R} \to [0, +\infty]$ with $\int f \, d\lambda < +\infty$. Consider the cumulative distribution function

$$F(x) := \int_{(-\infty, x]} f d\lambda.$$

Prove that F is continuous.

(c) Give an example of a Borel measure μ on the real line for which the cumulative distribution function

$$F(x) := \int_{(-\infty, x]} d\mu.$$

is *not* continuous.

Question 12

In this question, you will show why the lim inf in Fatou's lemma can't be naively replaced with a lim sup. Give an example of a measure space $(X, \mathcal{M}, \mu)$ and sequence of nonnegative measurable functions $\{f_n\}_{n\in\mathbb{N}}$ so that

$$\limsup_{n\to+\infty} \int f_n d\mu < \int \limsup_{n\to+\infty} f_n d\mu.$$

© Katy Craig, 2026