Question 2

Homework 5

(a) For every $a \in \mathbb{R}$, lower semicontinuity gives

$$f^{-1}((a, +\infty]) = \{x \in X : f(x) > a\} \in \mathcal{B}_X,$$

since this set is open. The rays $(a, +\infty]$ generate $\mathcal{B}_{\overline{\mathbb{R}}}$, so f is $(\mathcal{B}_X, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

(b) Define

$$q_h(x) = \frac{f(x+h) - f(x)}{h} \quad (h \neq 0), \qquad H_n(x) = \sup_{0<|h|<1/n} q_h(x).$$

For each fixed $h \neq 0$, the function q_h is continuous. By Homework 1, Question 8(a), the supremum H_n is lower semicontinuous, hence Borel measurable by part (a). By the definition of limsup,

$$g(x) = \lim_{n\to\infty} H_n(x) = \inf_{n\geq 1} H_n(x).$$

Thus g is the pointwise limit of a sequence of Borel measurable functions, so g is Borel measurable.

(c) In addition to the function g from part (b), define

$$\ell(x) = \liminf_{h\to 0} q_h(x) = -\limsup_{h\to 0}(-q_h(x)).$$

Applying part (b) to the continuous function $-f$ shows that ℓ is Borel measurable. Since $\ell \leq g$, density of the rationals gives

$$\{\ell < g\} = \bigcup_{a\in\mathbb{Q}} \big(\{\ell < a\} \cap \{g > a\}\big).$$

The right-hand side is a countable union of Borel sets. Therefore $E := \{\ell = g\} = \mathbb{R} \setminus \{\ell < g\}$ is Borel.

The function f is differentiable at x precisely when $q_h(x)$ tends to a finite real limit as $h \to 0$, equivalently when $\ell(x) = g(x) \in \mathbb{R}$. Hence

$$D = E \cap \{\ell > -\infty\} \cap \{g < +\infty\},$$

which is Borel because each of the three sets is Borel.

Question 3

Homework 5

(a)

For all $r>0$, $B(x,r)=\bigcup_{n=1}^{\infty} B(x,r-\frac{1}{n})$. Since $\{B(x,r-\frac{1}{n})\}_{n=1}^{\infty}$ is a sequence of nested sets, by continuity from below,
$\mu(B(x,r))=\lim_{n\to\infty}\mu(B(x,r-\frac{1}{n}))$. (*)

Consider a sequence $x_m\to x_0$. Then for all $n\in\mathbb{N}$, there exists M_n s.t.

$$|x_m-x_0|\le\frac{1}{n} \qquad \forall\ m\ge M_n$$

Thus $B(x_0,r-\frac{1}{n})\subseteq B(x_m,r)$ $\forall\ m\ge M_n$.
Therefore,

$$\liminf_{m\to+\infty}\mu(B(x_m,r)) \overset{\text{monotonicity}}{\ge} \mu(B(x_0,r-\tfrac{1}{n}))$$

Since $n\in\mathbb{N}$ was arbitrary, by (*), we obtain,

$$\liminf_{m\to+\infty}\mu(B(x_m,r))\ge\mu(B(x_0,r)).$$

This shows $x\mapsto\mu(B(x,r))$ is lowersemicts.

(b)

For all $r>0$, $\overline{B}(x,r) = \bigcap_{n=1}^{\infty} \overline{B}(x, r+\frac{1}{n})$.

By continuity from above (since μ is locally finite), $\overline{B}(x,r) = \lim_{n\to\infty} \overline{B}(x, r+\frac{1}{n})$.

Consider a sequence $x_m \to x_0$. As above, $\forall n \in \mathbb{N}$ $\exists M_n$ s.t.

$$\overline{B}(x_m, r) \subseteq \overline{B}(x_0, r+\tfrac{1}{n}) \quad \forall\, m \geq M_n.$$

As above,

$$\limsup_{m\to\infty} \mu(\overline{B}(x_m, r)) \leq \mu(\overline{B}(x_0, r+\tfrac{1}{n}))$$

and since $n \in \mathbb{N}$ was arbitrary,

$$\limsup_{m\to\infty} \mu(\overline{B}(x_m, r)) \leq \mu(\overline{B}(x_0, r)).$$

Thus, $x \mapsto \mu(\overline{B}(x,r))$ is upper semicontinuous.

(c)

By definition of limsup,

$$f(x) = \lim_{r \to 0} G_r(x).$$

By HW1, Q8, $G_r(x)$ is lower semicontinuous, hence Borel measurable.

Finally, $\lim_{r \to 0} G_r(x) = \lim_{n \to \infty} G_{1/n}(x)$, where the limit exists by monotonicity. Thus, $f(x)$ is a limit of measurable functions, hence measurable.

Question 4

Homework 5

For any $x_0 \in \mathbb{R}$, $1_{\{x_0\}}$ is Borel measurable, since $\{x_0\}$ is closed.
However, if $A \notin \mathcal{B}_{\mathbb{R}}$,

$$1_A(x) = \sup_{x_0 \in A} 1_{\{x_0\}}(x)$$

is not Borel measurable, since
$1_A^{-1}(\{1\}) = A \notin \mathcal{B}_{\mathbb{R}}$.

The chosen non-Borel set A is uncountable, since every countable subset of $\mathbb{R}$ is Borel. The functions $\{1_{\{x_0\}} : x_0 \in A\}$ are distinct, so this is an uncountable collection.

(a) By hypothesis, for all $n \in \mathbb{N}$,

$$n \, 1_{\{x : f(x) = +\infty\}}(x) \leq f(x)$$

Thus, $\forall n \in \mathbb{N}$,

$$\int n \, 1_{\{x : f(x) = +\infty\}} \, d\mu \leq \int f \, d\mu$$

$$n \, \mu(\{x : f(x) = +\infty\})$$

Since $\int f \, d\mu < +\infty$, this implies

$$\mu(\{x : f(x) = +\infty\}) = 0.$$

(b) Note that $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \frac{1}{n}\}$.

Furthermore,

$$\frac{1}{n} 1_{\{x : f(x) > \frac{1}{n}\}}(x) \leq f(x)$$

so $\frac{1}{n} \mu(\{x : f(x) > \frac{1}{n}\}) \leq \int f \, d\mu$.

This shows $\mu(\{x : f(x) > \frac{1}{n}\}) < +\infty \; \forall n \in \mathbb{N}$.

Thus $\{x : f(x) > 0\}$ is σ-finite.

(c) By part (a), $\{x : |f(x)| = +\infty\}$ is a null set.

Thus $f = f \, 1_{\{x : |f(x)| = +\infty\}^c}$ μ-a.e.,

and

$f \, 1_{\{x : |f(x)| = +\infty\}^c}$ is real valued.

^{c} denotes set complement

Question 8

Homework 5

By definition,

$$\int f d\mu = \sup\left\{\int \varphi d\mu : 0 \le \varphi \le f, \varphi \text{ simple}\right\}.$$

Thus, $\forall \varepsilon > 0$, $\exists$ φ_ε simple, $0 \le \varphi_\varepsilon \le f$, s.t. $\int \varphi_\varepsilon d\mu \ge \int f d\mu - \varepsilon$. Let $\varphi_\varepsilon = \sum_{i=1}^{n} c_i 1_{E_i}$, and suppose $c_i \neq 0$ $\forall i$.

Let $A = \bigcup_{i=1}^{n} E_i$, so $\int \varphi_\varepsilon 1_A d\mu = \int \varphi_\varepsilon d\mu$.

Furthermore, since $\varphi \le f$ and $f \in L^1(\mu)$, $+\infty > \int \varphi d\mu = \sum_{i=1}^{n} c_i \mu(E_i)$ and $\mu(E_i) < +\infty$ $\forall i$. Consequently, $\mu(A) < +\infty$.

Combining these facts, we obtain,

$$\int \varphi_\varepsilon 1_A d\mu = \int \varphi_\varepsilon d\mu \ge \int f d\mu - \varepsilon.$$

Thus,

$$\int f 1_A d\mu = \sup\left\{\int \varphi d\mu \; 0 \le \varphi \le f 1_A, \varphi \text{ simple}\right\}.$$

$$\ge \int \varphi_\varepsilon 1_A d\mu$$

$$\ge \int f d\mu - \varepsilon.$$

Question 10

Homework 5

Assume, for the sake of contradiction, that there exists $\varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ and $\int_E f \, d\mu \geq \varepsilon$.

In particular, choose a sequence $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ s.t. $\mu(E_n) < \frac{1}{2^n}$ and $\int_{E_n} f \, d\mu \geq \varepsilon$.

Consider the set $A = \bigcap_{n=1}^{\infty} \bigcup_{k>n} E_k$.

First, note that, by countable additivity, $\mu\left(\bigcup_{k>n} E_k\right) \leq \sum_{k>n} \mu(E_k) \leq \sum_{k>n} \frac{1}{2^k} = \frac{1}{2^n}$.

In particular $\mu\left(\bigcup_{k>1} E_k\right) \leq \frac{1}{2}$.

Since $B_n = \bigcup_{k>n} E_k$ satisfies $B_n \supseteq B_{n+1}$ and $\mu(B_1) < +\infty$, by continuity from above, $\mu(A) = \lim_{n\to\infty} \mu(B_n) \leq \lim_{n\to\infty} \frac{1}{2^n} = 0$.

On the other hand, since $B_n \supseteq E_{n+1}$ and $f \geq 0$

$$\int 1_{B_n} f \, d\mu \geq \int 1_{E_{n+1}} f \, d\mu \geq \epsilon.$$

Since $B_n \downarrow A$ and $f \geq 0$, we have

$$1_{B_n} f \downarrow 1_A f, \qquad \int 1_{B_1} f \, d\mu \leq \int f \, d\mu < +\infty.$$

Applying Question 9 and using $\mu(A) = 0$, we obtain

$$0 = \int 1_A f \, d\mu = \lim_{n\to\infty} \int 1_{B_n} f \, d\mu \geq \epsilon,$$

which is a contradiction.