Math 201A: Homework 6

Due Sunday, November 8, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1*

(a) Given a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable set E prove that

$$\int_E f(x)d\lambda(x) = \int_{E+c} f(x-c)d\lambda(x) \quad \text{for all } c \in \mathbb{R},$$

whenever the integral $\int_E f(x)d\lambda(x)$ exists.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function that is differentiable Lebesgue almost everywhere. (In fact, you can prove in a later measure theory course that all nondecreasing functions are differentiable Lebesgue almost everywhere!) Prove that

$$\int_{[0,1]} \limsup_{n\to+\infty} \frac{f(x+1/n)-f(x)}{1/n}\, d\lambda(x) \leq f(1)-f(0)$$

Hint: It can be convenient to assume, without loss of generality, that $f(x) = f(1)$ for all $x \geq 1$. Use part (a).

Question 2*

Prove the following *generalized dominated convergence theorem*:

THEOREM 1. *Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f_n, g_n, f, g \in L^1(\mu)$ are functions for which the following hold:*

(i) $f_n \to f$ μ*-a.e.* ,

(ii) $g_n \to g$ μ*-a.e.* ,

(iii) $|f_n| \leq g_n$ μ*-a.e.* ,

(iv) $\int g_n d\mu \to \int g d\mu$.

Then $\int f_n d\mu \to \int f d\mu$.

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Question 3*

Recall the definition of the Riemann integral from Homework 1, Question 4. Suppose that f is a bounded, real-valued function on $[a,b] \subseteq \mathbb{R}$. If it is Riemann integrable, we will denote its Riemann integral by $\int_a^b f(x)dx$.

(a) Prove that, if f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable on $[a,b]$, and

$$\int_a^b f(x)dx = \int_{[a,b]} f(x)d\lambda(x)$$

Hint:

(i) Define simple functions $u_P(x)$ and $l_P(x)$ so that $U(P,f) = \int u_P(x)d\lambda(x)$ and $L(P,f) = \int l_P(x)d\lambda(x)$.

(ii) Choose a sequence of partitions P_k that attains the infimum and the supremum in the definition of the upper and lower Riemann integrals, where P_{k+1} is a refinement of P_k for all $k \in \mathbb{N}$. Argue that this ensures $u_{P_k}(x)$ is decreasing while $l_{P_k}(x)$ is increasing and define their respective limits to be $u(x)$ and $l(x)$.

(iii) Explain why $l(x) \leq f(x) \leq u(x)$ for $x \in [a,b]$ and $\int u(x)d\lambda(x) = \int l(x)d\lambda(x) = \int_a^b f(x)dx$. Show this implies that $u = f$ almost everywhere in $[a,b]$.

(iv) Conclude that f is Lebesgue measurable. Use this to complete the proof.

(b) Consider the function $f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} 1_{(n,n+1]}(x)$. Since f is piecewise continuous, it is Riemann integrable on $[0,b]$ for all $b > 0$. Prove that $\lim_{b\to+\infty} \int_0^b f(x)dx$ exists, but $f \notin L^1(\lambda)$.

Question 4*

Consider the sequence of functions

$$f_n(x) = \frac{\sin(x/n)}{x/n}(1+x)^{-2} \quad \text{for } x \geq 0, \qquad f_n(x) = 0 \text{ for } x < 0.$$

We consider the expression $\sin(x)/x$ as an abbreviation for the function that takes the value $\sin(x)/x$ for $x \neq 0$ and 1 for $x = 0$.

(a) Are the functions f_n Borel measurable? Are they Lebesgue measurable? Justify your answers.

(b) Compute the following limit and justify your calculations. Explain where you use Question 3 and standard Calculus facts about the Riemann integral.

$$\lim_{n\to+\infty} \int_{[0,+\infty)} n\sin(x/n)\left[x(1+x)^2\right]^{-1} d\lambda(x)$$

Question 5

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $-\infty < a < b < +\infty$, and suppose $f : X \times [a,b] \to \mathbb{R}$ satisfies $f(\cdot,t) \in L^1(\mu)$ for every $t \in [a,b]$. Define

$$F(t) = \int_X f(x,t)\, d\mu(x).$$

Suppose there is a nonnegative $g \in L^1(\mu)$ such that $|f(x,t)| \leq g(x)$ for every $x \in X$ and $t \in [a,b]$. Prove that, if $f(x,\cdot)$ is continuous for each x, then F is continuous.

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Question 6*

Let $(X, \mathcal{M}, \mu)$ be a measure space, let $-\infty < a < b < +\infty$, and suppose $f : X \times [a, b] \to \mathbb{R}$ satisfies $f(\cdot, t) \in L^1(\mu)$ for every $t \in [a, b]$. Define $F(t) = \int_X f(x, t)\, d\mu(x)$. Suppose $\partial f/\partial t$ exists and

$$\left|\frac{\partial f}{\partial t}(x, t)\right| \leq g(x) \qquad \text{for every } x \in X,\ t \in [a, b],$$

where $g \in L^1(\mu)$ is nonnegative. (At a and b, derivatives are understood to be one-sided.)
In this problem, you will prove that F is differentiable and

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)\, d\mu(x). \tag{1}$$

Fix $t_0 \in [a, b]$. Let $\{t_n\} \subseteq [a, b] \setminus \{t_0\}$ be an arbitrary sequence converging to t_0, and define

$$h_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$

(a) Explain why each h_n is measurable and that $h_n(x) \to (\partial f/\partial t)(x, t_0)$ for every $x \in X$. Deduce that $x \mapsto (\partial f/\partial t)(x, t_0)$ is measurable and belongs to $L^1(\mu)$.

(b) Prove that $|h_n(x)| \leq g(x)$ for every $x \in X$ and $n \in \mathbb{N}$. *Hint: apply the mean value theorem.*

(c) Apply the dominated convergence theorem to show that

$$\lim_{n\to\infty} \int_X h_n(x)\, d\mu(x) = \int_X \frac{\partial f}{\partial t}(x, t_0)\, d\mu(x).$$

(d) Express $\int_X h_n\, d\mu$ as a difference quotient of F. Explain why the arbitrariness of $\{t_n\}$ proves the desired derivative formula (1).

Question 7

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n, f \in L^1(\mu)$ and $f_n \to f$ μ-almost everywhere. Prove that

$$\int_X |f_n - f|\, d\mu \longrightarrow 0 \quad \Longleftrightarrow \quad \int_X |f_n|\, d\mu \longrightarrow \int_X |f|\, d\mu.$$

Hint: use Question 2 for the reverse implication.

Question 8

Compute the following limit and justify your calculation:

$$\lim_{n\to\infty} \int_{[0,1]} \frac{1 + nx^2}{(1 + x^2)^n}\, d\lambda(x).$$

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