Question 1

Homework 6

First, suppose f is a simple function with standard representation $\sum_{i=1}^{n} c_i 1_{E_i}$. Then

$$f(x-c) = \sum_{i=1}^{n} c_i 1_{E_i}(x-c) = \sum_{i=1}^{n} c_i 1_{E_i+c}(x).$$

Furthermore

$$\int_E f = \sum_{i=1}^{n} c_i \lambda(E_i \cap E) = \sum_{i=1}^{n} c_i \lambda((E_i+c)\cap(E+c)) = \int_{E+c} f(x-c)\,d\lambda(x)$$

(translation invariance of Lebesgue measure)

Now, suppose f is a nonnegative, measurable function. Note that, for any simple fn φ,

$$0 \le \varphi(x) \le 1_E f(x) \iff 0 \le \varphi(x-c) \le 1_{E+c} f(x-c).$$

Thus,

$$\int_E f = \sup\{\int \varphi : 0 \le \varphi \le 1_E f,\ \varphi \text{ simple}\}$$

$$= \sup\{\int_E \varphi : 0 \le \varphi \le 1_E f,\ \varphi \text{ simple}\}$$

$$= \sup\left\{\int_{E+c} \varphi(x-c)\,d\lambda(x) : 0 \le \varphi(x-c) \le \underline{1}_{E+c}(x) f(x-c)\right\}$$

$$= \sup\left\{\int \varphi(x-c)\,d\lambda(x) : 0 \le \varphi(x-c) \le \underline{1}_{E+c}(x) f(x-c)\right\}$$

$$= \int_{E+c} f(x-c)\,d\lambda(x).$$

Finally, suppose $f: X \to \overline{\mathbb{R}}$ is measurable. Note that $f(x-c) = f_+(x-c) - f_-(x-c)$, so

$$\int_E f = \int_E f_+ - \int_E f_- = \int_{E+c} f_+(x-c)\,d\lambda(x) - \int_{E+c} f_-(x-c)\,d\lambda(x)$$

$$= \int_{E+c} f(x-c)\,d\lambda(x)$$

This completes the proof.

(b)

Since f is differentiable almost everywhere, for a.e. x,

$$\limsup_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} = \liminf_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} \in [0,+\infty)$$

Furthermore, if we define

$$g(x) = \begin{cases} f(x) & \text{for } x \in (-\infty, 1] \\ f(1) & \text{for } x \in (1,+\infty), \end{cases}$$

since $\lambda(\{1\}) = 0$, for a.e. $x \in [0,1]$,

$$\limsup_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} = \liminf_{n\to\infty} \frac{g(x+\frac{1}{n})-g(x)}{\frac{1}{n}}$$

Let N be the set of points in $[0,1]$ at which equality doesn't hold.

Note that, since g is real valued and nondecreasing, we have $g(x) \in [g(0), g(1)]$ for all $x \in [0,1]$, hence $g 1_{[0,1]} \in L^1(\lambda)$. Furthermore, as we will use below, $g(x + \frac{1}{n}) \in [g(0), g(1)]$ for all $x \in [0,1]$, $n \in \mathbb{N}$, so $g(x+\frac{1}{n}) 1_{[0,1]}(x) \in L^1(\lambda)$.

Combining these facts, we obtain,

$$\int_{[0,1]} \limsup_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} dx = \int_{[0,1]\setminus N} \text{"} \quad \text{"} + \underbrace{\int_N \text{"} \quad \text{"}}_{=0}$$

Fatou's Lemma

$$= \int \liminf_{n\to\infty} \underbrace{\frac{g(x+\frac{1}{n}) - g(x)}{\frac{1}{n}}}_{\geq 0 \text{ since } f \text{ is increasing}} 1_{[0,1]\setminus N} \, dx$$

$$\leq \liminf_{n\to\infty} \int \frac{g(x+\frac{1}{n}) - g(x)}{\frac{1}{n}} 1_{[0,1]\setminus N} \, dx$$

$$= \liminf_{n\to\infty} \int_{[0,1]} \text{“ } \text{ ” } - \int_{\mathbb{R}} \text{“ } \text{ ” }$$

$= 0$

linearity of integral

$$= \liminf_{n\to\infty} n\left(\int_{[0,1]} g(x+\tfrac{1}{n})\,dx - \int_{[0,1]} g(x)\,dx \right)$$

1(a)

$$= \liminf_{n\to\infty} n\left(\int_{[\frac{1}{n},1+\frac{1}{n}]} g - \int_{[0,1]} g \right)$$

$$\leq \liminf_{n\to\infty} n\left(\int_{[1,1+\frac{1}{n}]} g - \int_{[0,\frac{1}{n}]} g \right)$$

g is nondecreasing

$$\leq \liminf_{n\to\infty} n\left(\int_{[1,1+\frac{1}{n}]} g(1) - \int_{[0,\frac{1}{n}]} g(0) \right)$$

$$= \liminf_{n\to\infty} g(1) - g(0)$$

$$= g(1) - g(0)$$

defn of g

$$= f(1) - f(0)$$

Question 2

Homework 6

Following our proof of the Dominated Convergence Theorem from class,

Choose a subsequence so that $\lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n$.

Since $g_n - f_n \geq 0$ and $g_n + f_n \geq 0$ μ-a.e. $\forall n \in \mathbb{N}$, Fatou's Lemma ensures

$$\int g + \liminf_{n\to\infty} \int f_n = \lim_{k\to\infty} \int g_{n_k} + \lim_{k\to\infty} \int f_{n_k}$$

$$= \lim_{k\to\infty} \int g_{n_k} + \int f_{n_k} = \lim_{k\to\infty} \int g_{n_k} + f_{n_k}$$

$$\geq \int \liminf_{k\to\infty} g_{n_k} + f_{n_k} = \int g + f = \int g + \int f.$$

Similarly, we can show

$$\int g - \limsup_{n\to\infty} \int f_n \geq \int g - \int f$$

Subtracting $\int g$ from both sides gives

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n.$$

Thus, equality holds throughout, which gives the result.

Question 3

Homework 6

By definition,

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \quad , \quad M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \quad , \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

Set $u_P(a) = l_P(a) = f(a)$; for $x \in (a, b]$, define

$$u_P(x) = \sum_{i=1}^{n} M_i 1_{(x_{i-1},x_i]}(x), \qquad l_P(x) = \sum_{i=1}^{n} m_i 1_{(x_{i-1},x_i]}(x).$$

Extend both functions by zero outside $[a, b]$.

we have $U(P,f) = \int u_P d\lambda$, $L(P,f) = \int l_P d\lambda$.

Since f is Riemann integrable,

(*) $$\int_a^b f(x)dx = \inf_P U(P,f) = \sup_P L(P,f).$$

Recall that one partition P is a refinement of another partition $\tilde{P}$ if $\{x_0, x_1, \dots, x_n\} \supseteq \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{\tilde{n}}\}$.

Note that this implies

(**) $U(\tilde{P}, f) \geq U(P, f)$

$L(\tilde{P}, f) \leq L(P, f)$

Choose sequences of partitions P_k^U, P_k^L that attain the infimum and supremum in (*),

$\lim_{k\to\infty} U(P_k^U, f) = \inf_P U(P, f)$, $\lim_{k\to\infty} L(P_k^L, f) = \sup_P L(P, f)$.

By (**), refining a partition only makes the upper sum smaller and the lower sum bigger. Thus, we may define a new sequence P_k s.t. P_k is a refinement of P_k^U and P_k^L for all $k \in \mathbb{N}$, and

$\lim_{k\to\infty} U(P_k, f) = \inf_P U(P, f) = \sup_P L(P, f) = \lim_{k\to\infty} L(P_k, f)$.

Similarly, we may assume WLOG that P_{k+1} is a refinement of P_k for all $k \in \mathbb{N}$. This last fact ensures that $u_{P_k}(x)$ and $l_{P_k}(x)$ are decreasing/increasing, by definition of M_i and m_i. Since f is bounded, for each x, $\lim_{k\to\infty} u_{P_k}(x) =: u(x)$ and $\lim_{k\to\infty} l_{P_k}(x) =: l(x)$ exist and are real numbers.

Since $l_{P_k}(x) \le f(x) \le u_{P_k}(x)$ $\forall x \in [a,b]$, we have

(***) $l(x) \le f(x) \le u(x)$ $\forall x \in [a,b]$.

Furthermore,

$$\int u \, d\lambda \overset{\text{Dominated convergence theorem, since } |u_k| \le 1_{[a,b]}\|f\|_\infty \in L^1(\mu)}{=} \lim_{k\to\infty} \int u_k \, d\lambda = \lim_{k\to\infty} U(P_k, f) = \int_a^b f(x)\,dx$$

$$\int l \, d\lambda \overset{\text{Dominated convergence theorem, where } l_k \text{ has the same dominating fn as } u_k \text{ above}}{=} \lim_{k\to\infty} \int l_k \, d\lambda = \lim_{k\to\infty} L(P_k, f) = \int_a^b f(x)\,dx$$

In particular, we obtain $l \le u$, $\int l d\lambda = \int u d\lambda$, so $u - l \ge 0$ and $\int u - l \, d\lambda = 0$.
Thus, $u = l$ λ-a.e., so by (***), $u = f$ a.e. in $[a,b]$, and f is measurable.

Finally, by (***) and the fact $\int l d\lambda = \int u d\lambda$, we conclude,

$$\int_{[a,b]} f d\lambda = \int f 1_{[a,b]} d\lambda = \int u d\lambda = \int_a^b f(x) dx.$$

Lastly, since f is bounded and measurable, we have $\int_{[a,b]} |f| < +\infty$, so f is integrable on $[a,b]$.

b) By definition,

$$\int_0^b f(x) dx = \sum_{n=1}^{\lfloor b \rfloor - 1} \frac{(-1)^n}{n} + \frac{(-1)^{\lfloor b \rfloor}}{\lfloor b \rfloor} (b - \lfloor b \rfloor)$$

Thus, $\lim_{b \to \infty} \int_0^b f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} < +\infty$ by convergence of alternating series. On the other hand Beppo-Levi

$$\int |f| d\lambda = \sum_{n=1}^{\infty} \int \frac{1}{n} 1_{(n,n+1]} d\lambda = \sum_{i=1}^{\infty} \frac{1}{n} = +\infty,$$

so $f \notin L^1(\lambda)$.

Question 4

Homework 6

Define the continuous functions on $\mathbb{R}$

$$f_{1,n}(x) = \frac{\sin(x/n)}{x/n}, \qquad f_{2,n}(x) = \frac{1}{(1+\max\{x,0\})^2},$$

with $f_{1,n}(0) = 1$. Then $f_n = f_{1,n} f_{2,n} 1_{[0,+\infty)}$.

Both of these functions are cts, hence Borel measurable. $1_{[0,+\infty)}(x)$ is a simple fn, hence Borel measurable. Thus, the product of these three functions $fn(x)$ is Borel measurable. All Borel measurable functions are Lebesgue measurable.

(b) First, note that $\left|\frac{\sin(x)}{x}\right| \leq 1$.

Thus $|f_n(x)| \leq g(x) := \frac{1}{(1+x)^2} 1_{[0,+\infty)}(x)$.

Note that $g \in L^1(\lambda)$, since

$$\int g(x)\,d\lambda(x) = \int_{[0,+\infty)} \frac{1}{(1+x)^2} \overset{\text{MCT}}{=} \lim_{b\to+\infty} \int_{[0,b]} \frac{1}{(1+x)^2}\,d\lambda(x)$$

(corresp w/ Riemann integral)

$$= \lim_{b\to\infty} (-1)(1+x)^{-1}\Big|_0^b$$

$$= \lim_{b\to\infty} \left(1 - \frac{1}{1+b}\right)$$

$$= 1$$

Thus, by DCT,

$$\lim_{n\to\infty} \int f_n\,d\lambda = \int \lim_{n\to\infty} f_n\,d\lambda = \int_{[0,+\infty)} \frac{1}{(1+x)^2} = 1.$$

(by above computation for g & d)

The correspondence with the Riemann integral in the calculation above is Question 3(a), applied on each $[0, b]$; the antiderivative and the limit $\sin t/t \to 1$ are standard Calculus facts.

Question 6

Homework 6

Fix $t_0 \in [a,b]$ and an arbitrary sequence $t_n \in [a,b] \setminus \{t_0\}$ converging to t_0.

(a) Each h_n is a linear combination of the measurable functions $f(\cdot, t_n)$ and $f(\cdot, t_0)$, so it is measurable. By the definition of the derivative,

$$h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \longrightarrow \frac{\partial f}{\partial t}(x,t_0) \qquad (x \in X).$$

Thus $x \mapsto (\partial f/\partial t)(x,t_0)$ is measurable as a pointwise limit of measurable functions. The assumed bound gives

$$\int_X \left| \frac{\partial f}{\partial t}(x,t_0) \right| \, d\mu(x) \leq \int_X g \, d\mu < +\infty,$$

so this derivative belongs to $L^1(\mu)$.

(b) For fixed x and n, the mean value theorem applied to $t \mapsto f(x,t)$ on the interval with endpoints t_0, t_n gives a point $\xi_{x,n}$ strictly between them such that

$$h_n(x) = \frac{\partial f}{\partial t}(x, \xi_{x,n}).$$

Consequently $|h_n(x)| \leq g(x)$.

(c) By parts (a) and (b), the measurable functions h_n converge pointwise to $(\partial f/\partial t)(\cdot, t_0)$ and are dominated by $g \in L^1(\mu)$. The dominated convergence theorem therefore yields

$$\lim_{n\to\infty} \int_X h_n \, d\mu = \int_X \frac{\partial f}{\partial t}(x,t_0) \, d\mu(x).$$

(d) Linearity of the integral for $L^1(\mu)$ functions gives

$$\int_X h_n \, d\mu = \frac{\int_X f(x,t_n) \, d\mu(x) - \int_X f(x,t_0) \, d\mu(x)}{t_n - t_0} = \frac{F(t_n) - F(t_0)}{t_n - t_0}.$$

Since the sequence $\{t_n\}$ was arbitrary, the sequential characterization of limits proves that the difference quotient of F has a limit as $t \to t_0$ within $[a,b]$. Hence

$$F'(t_0) = \int_X \frac{\partial f}{\partial t}(x,t_0) \, d\mu(x).$$

This holds for every $t_0 \in [a,b]$, with the corresponding one-sided derivative when t_0 is an endpoint.