Math 201A: Homework 7

Due Sunday, November 22, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam. *All answers should be justified with either a proof or a counterexample.*

Question 1*

Consider a measure space $(X, \mathcal{M}, \mu)$. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Prove that $f_n + g_n \to f + g$ in measure.

(b) Prove that $f_n g_n \to fg$ in measure if $\mu(X) < +\infty$.

(c) Give a counterexample to show that the result of part (b) does not hold if $\mu(X) = +\infty$.

Question 2*

The goal of this problem is to prove Egoroff's Theorem.

THEOREM 1. *Consider a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < +\infty$. For any measurable, real-valued functions $\{f_n\}_{n=1}^{\infty}$, f, suppose that*

$$f_n \to f \ \mu\text{-almost everywhere.} \tag{1}$$

Then, for all $\epsilon > 0$, there exists $E \in \mathcal{M}$ so that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

Suppose the hypotheses of the theorem hold. Suppose, in addition, that there exist measurable, real-valued functions $\{g_n\}_{n=1}^{\infty}$, g so that $g_n(x) \to g(x)$ for all $x \in X$. Define

$$E_{n,k} = \cup_{m=n}^{\infty} \left\{ x : |g_m(x) - g(x)| \geq \frac{1}{k} \right\}.$$

(a) Prove that $\lim_{n \to +\infty} \mu(E_{n,k}) = 0$.

(b) Fix $\epsilon > 0$ arbitrary. Choose a subsequence n_k so that $\mu(E_{n_k,k}) < \epsilon/2^k$. Define $E = \cup_{k=1}^{\infty} E_{n_k,k}$. Prove that $\mu(E) < \epsilon$ and $g_n \to g$ uniformly on E^c.

(c) Use part (b) to complete the proof of the theorem.

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Question 3*

(i) Consider Lebesgue measure on $\mathbb{R}$ and the measurable functions $f_n = 1_{[n,n+1]}$ and $f = 0$. Show that $f_n \to f$ λ-almost everywhere. Furthermore, show that, there exists $\epsilon > 0$ so that, for all Lebesgue measurable sets E with $\lambda(E) < \epsilon$, f_n does not converge to f uniformly on E^c. Why is this example not a contradiction of Egoroff's theorem?

(ii) Consider Lebesgue measure restricted to $[0, 1]$ – that is, consider the measure defined by $\lambda_{[0,1]}(E) = \lambda(E \cap [0, 1])$ for all Lebesgue measurable sets E. Consider the measurable functions $f_n(x) = x^n$ and $f = 0$. Note that $f_n \to f$ $\lambda_{[0,1]}$-almost everywhere. Show that there is no Lebesgue measurable set E so that $\lambda_{[0,1]}(E) = 0$ and $f_n \to f$ uniformly on E^c. Why is this example not a contradiction of Egoroff's theorem?

Question 4

The goal of this question is to prove the following special case of *Lusin's theorem.*

THEOREM 2 (Lusin's Theorem)**.** *Consider a closed interval $[a, b] \subseteq \mathbb{R}$. Suppose $f : [a, b] \to \mathbb{R}$ is Lebesgue measurable and bounded. For all $\epsilon > 0$, there exists a compact set $E \subseteq [a, b]$ so that $\lambda([a, b] \setminus E) < \epsilon$ and $f|_E$ is continuous.*

(a) For any bounded, Lebesgue measurable function $f : [a, b] \to \mathbb{R}$, prove that there exists $\{g_k\}_{k=1}^{\infty} \subseteq C_c(\mathbb{R})$ so that $g_k \to f$ pointwise λ-a.e on $[a, b]$.

(b) For the sequence from part (a), prove that, for all $\epsilon > 0$, there exists a compact set $E \subseteq [a, b]$ so that $\lambda([a, b] \setminus E) < \epsilon$ and $g_k \to f$ uniformly on E.

(c) Apply the results of parts (a) and (b) to complete the proof of the theorem.

Hint: In order to apply Egoroff's theorem above, you will want to consider the restriction of Lebesgue measure to a set E, that is $\lambda_E(A) := \lambda(A \cap E)$.

Question 5

Parts (a) and (b) of this question are not related.

(a) Suppose $f_n \geq 0$ is a sequence of measurable functions satisfying $f_n \to f$ in measure. Prove that $\int f \leq \liminf_{n \to +\infty} \int f_n$.

(b) Consider a sequence of measurable sets E_n and suppose that $\mu(E_n) < +\infty$ for all $n \in \mathbb{N}$ and $1_{E_n} \to f$ in $L^1(\mu)$. Prove that there exists a measurable set E so that $f = 1_E$ μ-a.e..

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Question 6*

Consider a measure space $(X, \mathcal{M}, \mu)$. In class, we proved that if $f_n \to f$ in $L^1(\mu)$, then there is a subsequence f_{n_k} so that $f_{n_k} \to f$ μ-almost everywhere.

A natural question is whether this can be formulated as a statement along the lines of "convergent sequences in $L^1(\mu)$ are precompact in the μ-almost everywhere topology." Unfortunately, μ-almost everywhere convergence is not topological in general.

In this problem, you will prove that there is no topology that induces μ-almost everywhere convergence in general.

Recall the following fact: given a topological space $(X, \mathcal{T})$, $\{x_n\}_{n=1}^{\infty} \subseteq X$, and $x \in X$, we have that $x_n \to x$ if and only if every subsequence x_{n_k} has a further subsequence $x_{n_{k_l}}$ that converges to x.

(a) Give an example of a measure space and a sequence of measurable functions that converges in measure but not almost everywhere. Justify your example.

(b) Using the example from part (a), show that the equivalence in the recalled fact does not hold for almost everywhere convergence. Thus, almost everywhere convergence is not topological in general.

Question 7*

Let X be a nonempty set.

(a) Let $\mathcal{F}$ be a nonempty collection of monotone classes on X. Define

$$\cap\mathcal{F} = \{E : E \in \mathcal{C},\ \forall \mathcal{C} \in \mathcal{F}\}.$$

If $\cap\mathcal{F}$ is nonempty, prove that it is a monotone class.

(b) Given any nonempty $\mathcal{E} \subseteq 2^X$, prove that there is a smallest monotone class containing $\mathcal{E}$.

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Question 8*

(a) Consider $\mathcal{E}_\alpha \subseteq 2^{X_\alpha}$ nonempty, $\alpha \in A$, where the index set A is countable. Suppose $X_\alpha \in \mathcal{E}_\alpha$ and $\mathcal{M}_\alpha := \mathcal{M}(\mathcal{E}_\alpha)$. Prove that

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}\left(\{\Pi_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}\right).$$

(b) To see that $X_\alpha \in \mathcal{E}_\alpha$ is necessary, consider $A = \{1, 2\}$, $X_1 = X_2 = \{1, 2, 3\}$, $\mathcal{E}_1 = \{\{1\}\}$, and $\mathcal{E}_2 = \{\{2\}\}$. Explain why the equivalence proved in part (a) does not hold.

Hint for (a): Let $\pi_\alpha : \Pi_{\beta \in A} X_\beta \to X_\alpha$ denote the canonical projection. To show the containment $\subseteq$, explain why countability ensures that it suffices to show $\pi_\alpha^{-1}(E_\alpha)$ belongs to the set on the right hand side, for all $E_\alpha \in \mathcal{M}_\alpha$. Let $\mathcal{F}_\alpha$ denote the σ-algebra given by the pushforward of the σ-algebra on the right hand side under π_α, and show that $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$.

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