Lecture 10 - Math 201a

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HW5 due Monday, November 2nd

Thm: Given $f: X \to [0, +\infty]$ measureable, there exists a sequence fn of nonnegative simple fns s.t. $f_n \nearrow f$ pointwise.

Thm (Beppo-Levi) Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Thm (Fatou's Lemma):
Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable,

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu$$

Important Examples of Inequality

$(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$

Ex: ("running away to infinity")

$f_n := 1_{[n, n+1]}$, $\lim_{n\to\infty} f_n = 0$

Ex: ("going up the spout")

$f_n := n\, 1_{[0, \frac{1}{n}]}$, $\lim_{n\to\infty} f_n(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

$$Q: \sum_{n=1}^{\infty} \int |f_n - f_{n+1}| d\mu < +\infty \quad \boxed{?}$$

Our second example from last time relied on the following fact:

Prop: Given $f: X \to [0, +\infty]$ measureable,

$$\int f d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}$$

$$\mu(\{x : f(x) \neq 0\}) = 0$$

Pf: First, suppose f is simple, i.e.

$$f = \sum_{i=1}^{n} a_i 1_{E_i}, \quad a_i \geq 0.$$

Then,
$$0 = \int f \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$$

$\Leftrightarrow \forall i$, either $a_i = 0$ or $\mu(E_i) = 0$.

$\Leftrightarrow f = 0$ μ-a.e.

Now, suppose f is nonnegative, meas fn. Then

$$\int f \, d\mu = \sup \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\}$$

To see "$\Leftarrow$", note that if $f = 0$ μ-a.e., then for all φ in constraint set, $\varphi = 0$ μ-a.e. Thus $\int \varphi \, d\mu = 0$, so $\int f \, d\mu = 0$.

To see "$\Rightarrow$", suppose $\mu(\{x : f(x) \neq 0\}) > 0$.
We will show $\int f \, d\mu > 0$.

Note that

$$0 < \mu(\{x : f(x) \neq 0\}) = \mu\left(\bigcup_{n=1}^{\infty} \{x : f(x) \geq \tfrac{1}{n}\}\right)$$

$$\leq \sum_{n=1}^{\infty} \mu(\{x : f(x) \geq \tfrac{1}{n}\})$$

Thus, $\exists\, n \in \mathbb{N}$ s.t. $\mu(\{x : f(x) \geq \tfrac{1}{n}\}) > 0$.

Therefore:

$$\int f \, d\mu \geq \int_{\{x : f(x) \geq \frac{1}{n}\}} f \, d\mu \geq \int_{\{x : f(x) \geq \frac{1}{n}\}} \left(\tfrac{1}{n}\right) d\mu$$

$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$

$$\left(\tfrac{1}{n}\right) \mu(\{x : f(x) \geq \tfrac{1}{n}\}) > 0$$

Integration of Real Valued Fns

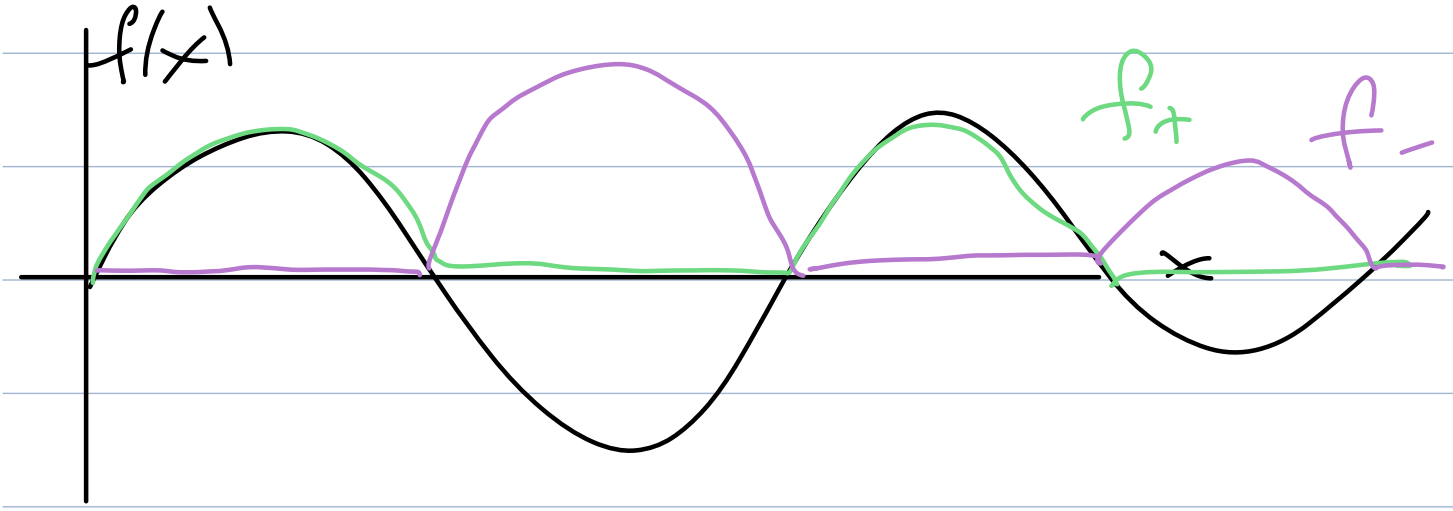
Measure space $(X, \mathcal{M}, \mu)$

Given $f: X \to \overline{\mathbb{R}}$, define $(f \vee 0)(x) := \max\{f(x), 0\}$

"positive part" $f_+ := f \vee 0$

"negative part" $f_- := (-f) \vee 0$

Thus, $f = f_+ - f_-$ & $|f| = f_+ + f_-$



Def: Given $f: X \to \overline{\mathbb{R}}$ measurable, if either $\int f_+ d\mu < +\infty$ or $\int f_- d\mu < +\infty$, then we define

"the integral exists"

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu$$

If both $\int f_+ d\mu < +\infty$ and $\int f_- d\mu < +\infty$ then we say f is integrable and write $f \in L^1(\mu)$.

$$\int f_+ + f_- \, d\mu < +\infty = \int f_+ d\mu + \int f_- d\mu < +\infty$$

$$\Updownarrow$$

$$\int |f| d\mu < +\infty$$

To deal with the cases of $\pm\infty$...

Prop: If $f \in L^1(\mu)$, then

$$\left| \int f d\mu \right| \leq \int |f| d\mu$$

Pf: $\left|\int f\,d\mu\right| = \left|\int f_+\,d\mu - \int f_-\,d\mu\right|$

$\leq \int f_+\,d\mu + \int f_-\,d\mu$

$= \int f_+ + f_-\,d\mu$

$= \int |f|\,d\mu$ □

Cor: If $f, g \in L^1(\mu)$, then

$\int |f-g|\,d\mu = 0 \iff f = g$ μ-a.e.

Pf: Since $\psi := |f-g|$ is a nonneg. meas fn, by earlier prop $\int \psi\,d\mu = 0 \iff \psi = 0$ μ-a.e.

$\{x : \psi(x) = 0\} = \{x : |f(x) - g(x)| = 0\}$

$= \{x : f(x) = g(x)\}$ □

Remark: If $f: X \to \overline{\mathbb{R}}$ satisfies $f \in L^1(\mathbb{R})$, HW5 Q7 ensures $\exists\ \tilde{f}: X \to \mathbb{R}$ s.t. $f = \tilde{f}$ μ-a.e.

fix intermediate argument :

$$\int f_+ d\mu = \int \tilde{f}_+ d\mu, \quad \int f_- d\mu = \int \tilde{f}_- d\mu.$$

Prop: $L^1(\mu)$ is a real vector space and

$$f \mapsto \int f \, d\mu$$

is a linear functional on $L^1(\mu)$.

Pf: To show $L^1(\mu)$ is a real vector space, fix $f, g \in L^1(\mu)$ and $a, b \in \mathbb{R}$. WTS $af + bg \in L^1(\mu)$.

If $f, g : X \to \mathbb{R}$,

$$\int |af + bg| \, d\mu \leq \int |a||f| + |b||g| \, d\mu$$

$$\leq |a| \int |f| \, d\mu + |b| \int |g| \, d\mu$$

$$< +\infty,$$

so $af + bg \in L^1(\mu)$.

Next, fix $f \in L^1(\mu)$, $a \geq 0$. Then

$$\begin{aligned}\int af\,d\mu &= \int af^+\,d\mu - \int af^-\,d\mu \\ &= a\int f^+\,d\mu - a\int f^-\,d\mu \\ &= a\int f\,d\mu\end{aligned}$$

For $a < 0$, note that $af = (-a)(-f)$, and apply previous fact to $-f$:

$$\begin{aligned}\int af\,d\mu = \int (-a)(-f)\,d\mu &= (-a)\int (-f)\,d\mu \\ &= a\int f\,d\mu.\end{aligned}$$

Finally, fix $f, g \in L^1(\mu)$.

Suppose $f, g : X \to \mathbb{R}$.

$$\begin{aligned}\int (f+g)\,d\mu &= \int (f+g)^+\,d\mu - \int (f+g)^-\,d\mu \\ &\overset{\text{CLAIM}}{=} \int f^+\,d\mu + \int g^+\,d\mu - \int f^-\,d\mu - \int g^-\,d\mu \\ &= \int f\,d\mu + \int g\,d\mu\end{aligned}$$

Justification of CLAIM:

$$(f+g)_+ - (f+g)_- = (f_+ - f_-) + (g_+ - g_-)$$

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$$

By the integral of sum of nonneg rdcas fns...

$$\int (f+g)_+ d\mu + \int f_- d\mu + \int g_- d\mu$$

$$= \int (f+g)_- d\mu + \int f_+ d\mu + \int g_+ d\mu$$

rearranging gives the CLAIM. □

Thm (Dominated convergence)

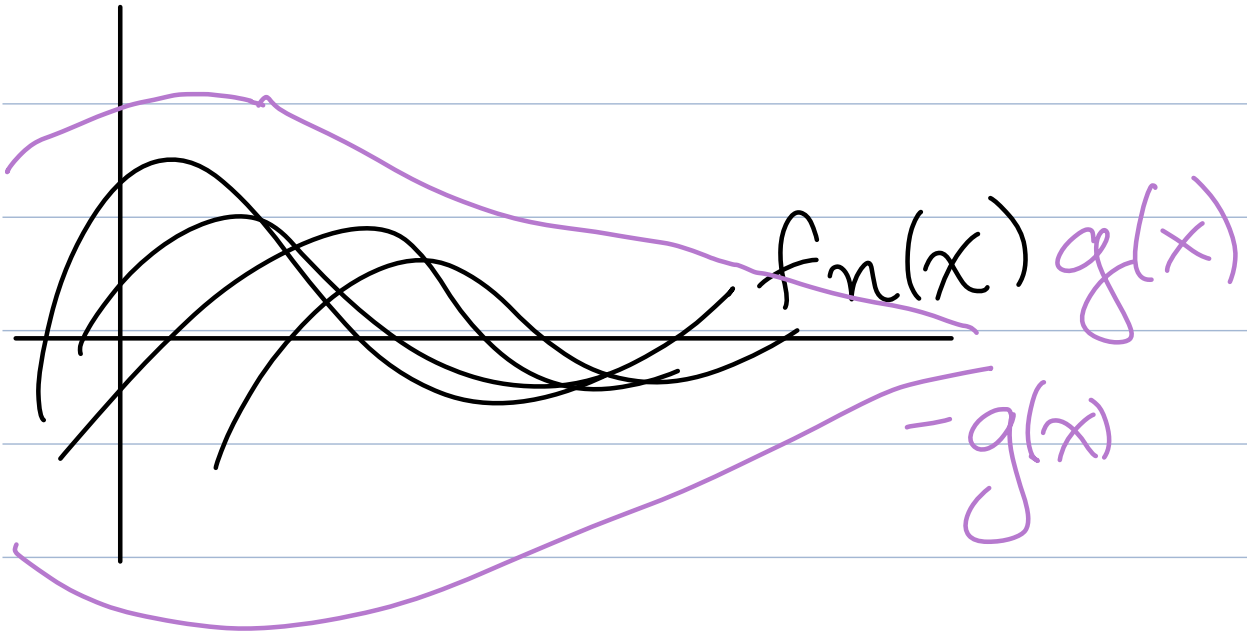
Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ s.t. $\lim_{n\to\infty} f_n(x)$ $\forall x \in X$ and $\exists\, g \in L^1(\mu)$ s.t.

$$\Rightarrow -g \leq f_n \leq g$$

$$|f_n| \leq g \quad \forall n \in \mathbb{N}$$

we have that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$



Rmk: $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$

$g \in L^1(\lambda)$

g is bounded

$\exists M > 0$ s.t. $|g(x)| \leq M \ \forall x \in X$

$\not\Leftarrow$ since $g(x) \equiv 1$ satisfies $\int g(x) \, d\lambda(x) = +\infty$

$\not\Rightarrow$ $g(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

will show equiv w/ Riemann int

$$\int_{\mathbb{R}} g(x) \, d\lambda(x) = \int_0^1 \frac{1}{\sqrt{x}} \, dx = 2 < +\infty$$