Lecture 11 - Math 201a

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- Midterm 2 - Thursday, Nov 12th
- no HW next week, just practice mid
- Extra office hours Mon, Nov 9th 3:15-4:15pm

Prop: Given $f: X \to [0, +\infty]$ measureable,

$$\int f \, d\mu = 0 \iff f = 0 \ \mu\text{-a.e.}$$

$$\mu(\{x : f(x) \neq 0\}) = 0$$

Given $f: X \to \overline{\mathbb{R}}$, define

"positive part" $f_+ := f \vee 0$

"negative part" $f_- := (-f) \vee 0$

Thus, $f = f_+ - f_-$ & $|f| = f_+ + f_-$

Def: Given $f: X \to \overline{\mathbb{R}}$ measurable, if either $\int f_+ d\mu < +\infty$ or $\int f_- d\mu < +\infty$, then we define

("the integral exists")

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu$$

If both $\int f_+ d\mu < +\infty$ and $\int f_- d\mu < +\infty$ then we say f is integrable and write $f \in L^1(\mu)$.

$$\int f_+ + f_- \, d\mu < +\infty = \int f_+ d\mu + \int f_- d\mu < +\infty$$

$$\Updownarrow$$

$$\int |f| d\mu < +\infty$$

Remark: If $f: X \to \overline{\mathbb{R}}$ satisfies $f \in L^1(\mathbb{R})$, HW5 Q7 ensures $\exists\ \tilde{f}: X \to \mathbb{R}$ s.t. $f = \tilde{f}$ μ-a.e.

Remark: If $f = g$ μ-a.e., then

- $f_+ = g_+$ μ-a.e.

$\{x : f_+(x) \neq g_+(x)\} \subseteq \{x : f(x) \neq g(x)\}$

this has μ-measure 0

- $f_- = g_-$ μ-a.e.

- $\int f_+ \, d\mu = \int f_+ (1_E + 1_{E^c}) \, d\mu$

$E := \{x : f_+(x) = g_+(x)\}$

$= \int f_+ 1_E \, d\mu + \int f_+ 1_{E^c} \, d\mu$

$\int f_+ 1_{E^c} \, d\mu = 0$ ($= 0$ μ-a.e.)

$= \int g_+ 1_E \, d\mu + \int g_+ 1_{E^c} \, d\mu$

$\int g_+ 1_{E^c} \, d\mu = 0$

$= \int g_+ \, d\mu$

- $\int f_- \, d\mu = \int g_- \, d\mu$

Thus, $\int |f| \, d\mu = \int |g| \, d\mu$.

Thus, if the integral of either f or g exists, $\int f \, d\mu = \int g \, d\mu$

Prop: $L^1(\mu)$ is a real vector space and

$$f \mapsto \int f\,d\mu$$

is a linear functional on $L^1(\mu)$.

Pf: To show $L^1(\mu)$ is a real vector space, fix $f, g \in L^1(\mu)$ and $a, b \in \mathbb{R}$. WTS $af + bg \in L^1(\mu)$.

If $f, g : X \to \mathbb{R}$,

$$\int |af + bg|\,d\mu \leq \int |a||f| + |b||g|\,d\mu$$
$$\leq |a| \int |f|\,d\mu + |b| \int |g|\,d\mu$$
$$< +\infty,$$

so $af + bg \in L^1(\mu)$.

* may be undefined on a set of meas. zero

If $f, g : X \to \overline{\mathbb{R}}$, applying this to $\tilde{f}, \tilde{g}$, we see $a\tilde{f} + b\tilde{g} \in L^1(\mu)$. Thus, since * $af + bg = a\tilde{f} + b\tilde{g}$ μ-a.e., so $\int |af + bg|\,d\mu = \int |a\tilde{f} + b\tilde{g}|\,d\mu < +\infty$.

Next, fix $f \in L^1(\mu)$, $a \geq 0$. Then

$$\begin{aligned}\int af\,d\mu &= \int af^+\,d\mu - \int af^-\,d\mu \\ &= a\int f^+\,d\mu - a\int f^-\,d\mu \\ &= a\int f\,d\mu\end{aligned}$$

For $a < 0$, note that $af = (-a)(-f)$, and apply previous fact to $-f$:

$$\begin{aligned}\int af\,d\mu = \int (-a)(-f)\,d\mu &= (-a)\int (-f)\,d\mu \\ &= a\int f\,d\mu.\end{aligned}$$

Finally, fix $f, g \in L^1(\mu)$.

Suppose $f, g : X \to \mathbb{R}$.

CLAIM

$$\begin{aligned}\int (f+g)\,d\mu &= \int (f+g)^+\,d\mu - \int (f+g)^-\,d\mu \\ &= \int f^+\,d\mu + \int g^+\,d\mu - \int f^-\,d\mu - \int g^-\,d\mu \\ &= \int f\,d\mu + \int g\,d\mu\end{aligned}$$

Justification of CLAIM:

$$(f+g)_+ - (f+g)_- = (f_+ - f_-) + (g_+ - g_-)$$

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+$$

By the integral of sum of nonneg rdeas fns...

$$\int (f+g)_+ d\mu + \int f_- d\mu + \int g_- d\mu$$

$$= \int (f+g)_- d\mu + \int f_+ d\mu + \int g_+ d\mu$$

rearranging gives the CLAIM

More generally, if $f, g : X \to \overline{\mathbb{R}}$, applying this argument to $\tilde{f}, \tilde{g}$ gives

$$\int (f+g) d\mu = \int (\tilde{f}+\tilde{g}) d\mu = \int \tilde{f} d\mu + \int \tilde{g} d\mu$$

$$= \int f d\mu + \int g d\mu$$

□

Prop: If $f \in L^1(\mu)$, then
$\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$

Cor: If $f, g \in L^1(\mu)$, then

$\int |f-g| \, d\mu = 0 \iff f = g$ μ-a.e.

We wish we could show

$$L^1(\mu) = \{f : X \to \overline{\mathbb{R}} : \int |f| \, d\mu < +\infty\}$$

is a normed vector space with

$$\|f\|_{L^1(\mu)} := \int |f| \, d\mu.$$

Problem: this norm would be degenerate, since $\exists f \in L^1(\mu)$ with $f \neq 0$, but $\|f\|_{L^1(\mu)} = 0$.

$\exists x$ s.t. $f(x) \neq 0$.

e.g. $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

Easy fix:

Def: $L^1(\mu) := \{f : X \to \overline{\mathbb{R}} : \int |f| d\mu < +\infty\} / \sim$

where $f \sim g$ iff $f = g$ μ-a.e.

Remark: By abuse of notation, we will let $f \in L^1(\mu)$ denote

1. the equivalence class
2. a representative of equiv class

$$x^2 \cdot 1_{[0,1]} \in L^1(\mu)$$

vs

$$x^2 \cdot 1_{[0,1]} + 1_{\{0\}}$$

3. a representative that is only defined μ-a.e.

$$\frac{1}{\sqrt{x}} 1_{[0,1)} - \frac{2}{x^{3/4}} 1_{[0,1]}.$$

Prop: $\|f\|_{L^1(\mu)} := \int |f| d\mu$ is a norm on $L^1(\mu)$.

Pf: nondegeneracy ✓

triangle ineq ✓

$$\|f+g\| = \int |f+g| d\mu \leq \int |f| + |g| d\mu = \|f\| + \|g\|$$

positive homogeneity ✓

$$\|af\| = \int |af| d\mu = \int |a||f| d\mu = |a| \|f\| \square$$

Now... properties of convergence in this norm...

When does $\int |f_n - f| d\mu \xrightarrow{n\to\infty} 0$?

Key tool...

Thm (Dominated convergence)

Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ s.t.

$\lim_{n\to\infty} f_n(x)$ exists μ-a.e.

$\exists\, g \in L^1(\mu)$ s.t.

$$|f_n| \leq g \quad \mu\text{-a.e.} \quad \forall n \in \mathbb{N}$$

we have that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

Rmk: $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$

$g \in L^1(\lambda)$ $\not\Longrightarrow$ g is bounded

Pf: Since $\lim_{n\to\infty} f_n$ exists μ-a.e. and $|f_n| \le g$ μ-a.e. $\forall n \in \mathbb{N}$, we have

$$\left|\lim_{n\to\infty} f_n\right| \le g \quad \mu\text{-a.e.}$$

$$\{x : \left|\lim_{n\to\infty} f_n(x)\right| > g(x) \text{ or } \lim_{n\to\infty} f_n(x) \text{ D.N.E}\}$$
$$\subseteq B \cup \left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

by countable subadditivity, LHS has μ-measure zero.

$$\{x : |f_n(x)| > g(x)\} =: A_n$$

$$\{x : \lim_{n\to\infty} f_n(x) \text{ D.N.E}\} =: B$$

Since $g \in L^1(\mu)$, $\lim_{n\to\infty} f_n \in L^1(\mu)$.

Since $|f_n| \le g$ μ-a.e., so

$g - f_n \ge 0$, $g + f_n \ge 0$ μ-a.e.

By Fatou's lemma,

$-g \le f_n \le g$

$$\int g\,d\mu - \limsup_{n\to\infty} \int f_n\,d\mu$$

$$\liminf_{n\to\infty} \int g\,d\mu - \int f_n\,d\mu$$

$\liminf_{n\to\infty} -z_n = -\limsup_{n\to\infty} z_n$

$$\liminf_{n\to\infty} \int (g - f_n)\,d\mu \ge \int \liminf_{n\to\infty} g - f_n\,d\mu$$

$$= \int g - \lim_{n\to\infty} f_n\,d\mu$$

linearity of integral 27(a)

$$= \int g\,d\mu - \int \lim_{n\to\infty} f_n\,d\mu$$

Again, by Fatou,

$$\int g\,d\mu + \liminf_{n\to\infty} \int f_n\,d\mu$$

$$\liminf_{n\to\infty} \int g + f_n\,d\mu \ge \int \liminf_{n\to\infty} g + f_n\,d\mu$$

$$= \int g\,d\mu + \int \lim_{n\to\infty} f\,d\mu$$

Combining these,

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \lim_{n\to\infty} f_n \, d\mu$$

$$\| $$

$$\limsup_{n\to\infty} \int f_n \, d\mu$$

Therefore,

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu. \qquad \square$$

We now apply DCT to identify two useful classes of functions that are dense in $L^1(\mu)$.

Thm:

[A] For any measure space $(X, \mathcal{M}, \mu)$, simple functions are dense in $L^1(\mu)$.

[B] For a Lebesgue-Stieltjes measure μ on $\mathbb{R}$, the following are also dense in $L^1(\mu)$:

(i) simple functions of the form $\xi = \sum_{j=1}^{N} a_j 1_{F_j}$, $F_j = \bigcup_{i=1}^{m_j} I_{ij}$

for disjoint open intervals $\{I_{ij}\}_{i=1}^{m_j}$

(ii) $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is cts and } \operatorname{supp} f \text{ is cpt}\}$

$\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$