Lecture 12 - Math 201a

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- Midterm 2 - Thursday, Nov 12th
- no HW next week, just practice mid
- Extra office hours Mon, Nov 9th 3:15-4:15pm

Remark: If $f = g$ μ-a.e., then

$$\int f_+ \, d\mu = \int g_+ \, d\mu$$
$$\int f_- \, d\mu = \int g_- \, d\mu$$

Thus, $\int |f| \, d\mu = \int |g| \, d\mu$.

Thus, if the integral of either f or g exists, $\int f \, d\mu = \int g \, d\mu$

Def: $L^1(\mu) := \{f : X \to \overline{\mathbb{R}} : \int |f| d\mu < +\infty\} / \sim$

where $f \sim g$ iff $f = g$ μ-a.e.

Prop: $\|f\|_{L^1(\mu)} := \int |f| d\mu$ is a norm on $L^1(\mu)$.

Thm (Dominated convergence)
Given $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ s.t.
$\lim_{n\to\infty} f_n(x)$ exists μ-a.e.
$\exists\, g \in L^1(\mu)$ s.t.

$$|f_n| \leq g \quad \mu\text{-a.e.} \quad \forall n \in \mathbb{N}$$

we have that

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu.$$

Thm:

[A] For any measure space $(X, \mathcal{M}, \mu)$, simple functions are dense in $L^1(\mu)$.

[B] For a Lebesgue-Stieltjes measure μ on $\mathbb{R}$, the following are also dense in $L^1(\mu)$:

(i) simple functions of the form $\xi = \sum_{j=1}^{N} a_j 1_{F_j}$, $F_j = \bigcup_{i=1}^{m_j} I_{ij}$

for disjoint open intervals $\{I_{ij}\}_{i=1}^{m_j}$

(ii) $C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is cts and } \operatorname{supp} f \text{ is cpt}\}$

$\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$

Pf: Fix $f \in L^1(\mu)$. Since f_+, f_- are nonneg, meas fns, $\exists$ simple fns φ_n, ψ_n s.t.

$$\varphi_n \nearrow f_+, \quad \psi_n \nearrow f_-.$$

Thus,

$$|(\varphi_n - \psi_n) - f| \xrightarrow{n\to\infty} 0$$

$$|(\varphi_n - \psi_n) - f| \le \varphi_n + \psi_n + |f| \le 2|f|$$

By DCT,

$$\lim_{n\to\infty} \int |\underbrace{(\varphi_n - \psi_n)}_{\text{simple function}} - f| \, d\mu = 0.$$

This shows A.

Now show (F). Fix $\phi \in L^1(\mu)$ simple. It suffices to show we may approximate ϕ arb well in $L^1(\mu)$ by "even simpler" fns in (i). ☺

Express $\phi = \sum_{j=1}^{n} a_j 1_{E_j}$.

WLOG $a_j \neq 0 \ \forall j$, $\{E_j\}_{j=1}^n$ disjoint.

Since $\phi \in L^1(\mu)$,

$$+\infty > \int |\phi| \, d\mu \geq |a_j| \, \mu(E_j) \Rightarrow \mu(E_j) < +\infty \quad (\forall j)$$

$$\geq \int_{E_j} |\phi| \, d\mu$$

By Fact from HW, $\forall j$, $\exists \{I_j^i\}_{i=1}^{m_j}$ disjoint open intervals s.t.

$$\mu\left(E_j \triangle \bigcup_{i=1}^{m_j} I_j^i\right) < \frac{\varepsilon}{n \max_j |a_j|}$$

Fact (HW4, Q7):
Given a Lebesgue-Stieltjes measure μ_F, for any $E \in \mathcal{M}_{\mu_F}$ with $\mu_F(E) < +\infty$ and for all $\varepsilon > 0$, there exist disjoint open intervals $\{I^i\}_{i=1}^{N}$ s.t.

$$\mu_F\left(E \triangle \bigcup_{i=1}^{N} I^i\right) < \varepsilon.$$

Then,

$$\left\| \phi - \sum_{j=1}^{n} a_j \mathbf{1}_{\bigcup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)}$$

$$\leq \sum_{j=1}^{n} |a_j| \underbrace{\left\| \mathbf{1}_{E_j} - \mathbf{1}_{\bigcup_{i=1}^{m_j} I_j^i} \right\|_{L^1(\mu)}}_{\mu\left(E_j \triangle \bigcup_{i=1}^{m_j} I_j^i\right)}$$

$$\leq \varepsilon$$

This shows B(i).

Finally, we show B (ii).

Fix an "even simpler" fn

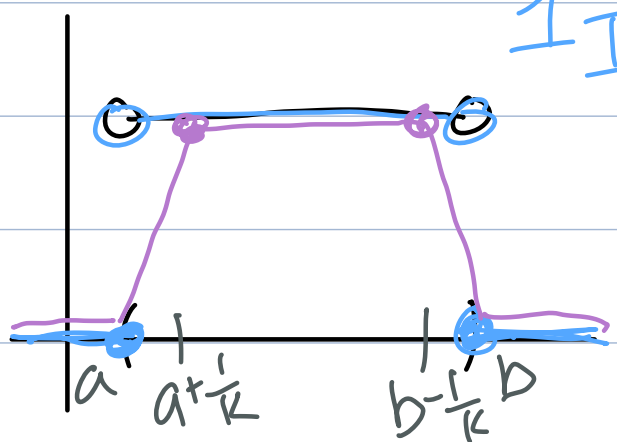
$$\sum_{j=1}^{n} a_j \mathbb{1}_{\bigcup_{i=1}^{m_j} I_j^i},$$

where, for each j, $\{I_j^i\}_{i=1}^{m_j}$ are disjoint open intervals.

We will show this may be approximated arbitrarily well by a cts, cptly supported fn.

Claim: For any open interval I, $\varepsilon > 0$, $\exists\, f \in C_c(\mathbb{R})$ s.t. $\|f - \mathbb{1}_I\|_{L^1(\mu)} < \varepsilon$.

Pf:



$$\|f - 1_I\|_{L^1(\mu)} = \int_{(a,b)} |f - 1_I| \, d\mu$$

$$= \int_{(a,a+1/k)} |f-1| \, d\mu + \int_{(b-1/k,\,b)} |f-1| \, d\mu$$

$$\leq \mu((a,a+1/k)) + \mu((b-1/k,\,b))$$

$$\xrightarrow{k\to\infty} \mu\Big(\bigcap_{k\in\mathbb{N}} (a,a+1/k)\Big) + \mu\Big(\bigcap_{k\in\mathbb{N}} (b-1/k,\,b)\Big)$$

$$= \mu(\emptyset) = 0.$$

Thus, $\forall i,j$, choose $f_j^i \in C_c(\mathbb{R})$ s.t. $\|f - f_{I_j}^i\|_{L^1(\mu)} < \dfrac{\varepsilon}{n \max_j |a_j| \max_j m_j}$

Then, this is a cts, cptly supported function

$$\Big\| \sum_{j=1}^{n} a_j 1_{\bigcup_{i=1}^{m_j} I_j^i} - \sum_{j=1}^{n} \sum_{i=1}^{m_j} a_j f_j^i \Big\|_{L^1(\mu)}$$

$$= \Big\| \sum_{j=1}^{n} \sum_{i=1}^{m_j} a_j 1_{I_j^i} - \sum_{j=1}^{n} \sum_{i=1}^{m_j} a_j f_j^i \Big\|_{L^1(\mu)}$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{m_j} |a_j| \, \| 1_{I_j^i} - f_j^i \|_{L^1(\mu)} < \varepsilon. \quad \square$$

Modes of Convergence

$(X, \mathcal{M}, \mu)$, $f_n, f : X \to \mathbb{R}$ meas

① uniform convergence

$\Downarrow$ $\sup_{x \in X} |f_n(x) - f(x)| \to 0$

② pointwise convergence

$f_n(x) \to f(x) \quad \forall x \in X$

$\Downarrow$

③ pointwise a.e. convergence

$f_n(x) \to f(x) \quad \mu\text{-a.e. } x \in X$

④ L^1 convergence, $\|f_n - f\|_{L^1(\mu)} \to 0$

How are ③ and ④ related?

Ex: ① $\not\Rightarrow$ ④

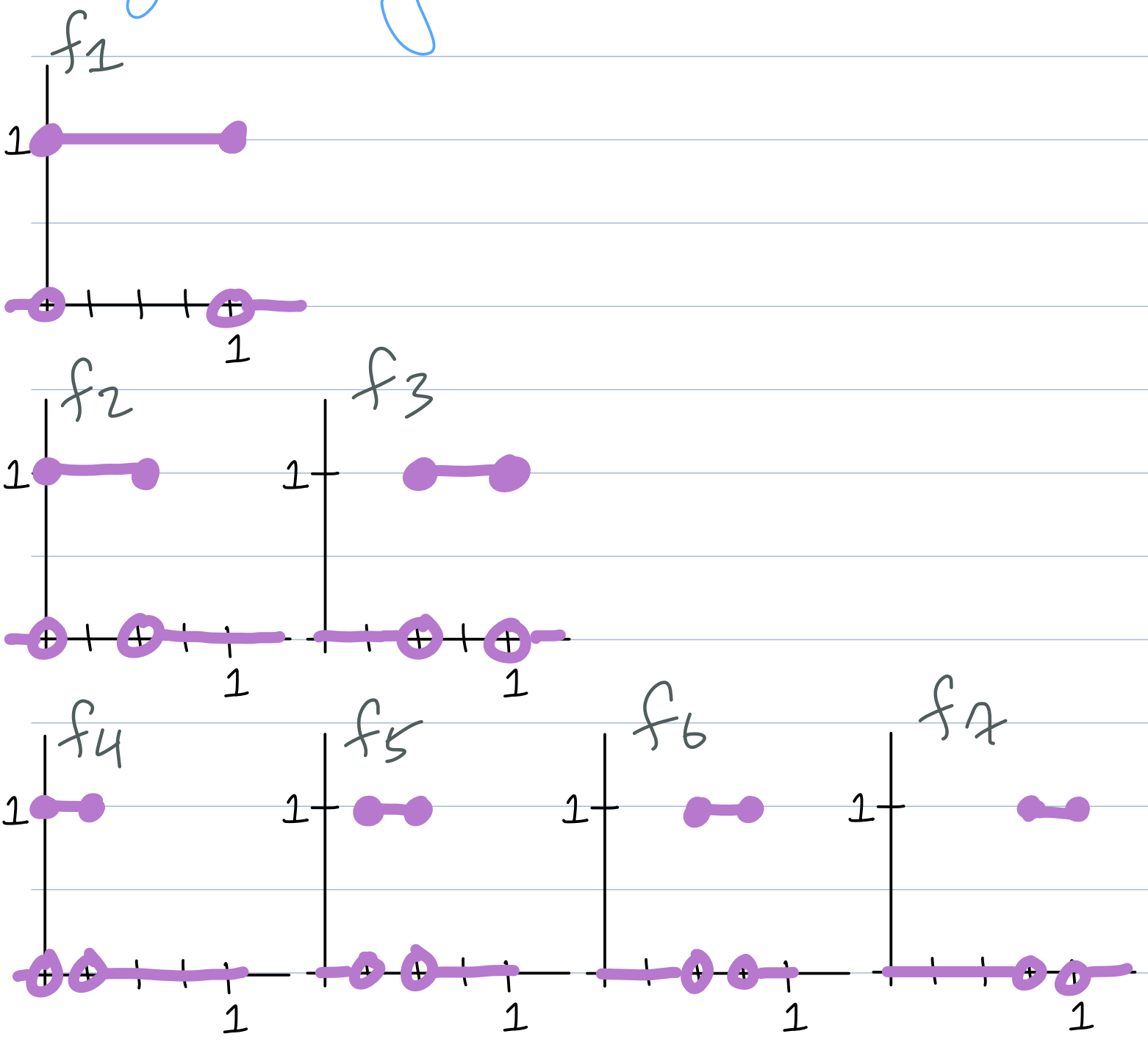
$f_n(x) = \frac{1}{n} 1_{[0,n]}$ $\quad f = 0$

$f_n \to f$ uniformly

$\|f_n - f\|_{L^2(\lambda)} = 1$

Ex: ④ $\not\Rightarrow$ ③

Refining Wave



$f_n \to 0$ in $L^1(\lambda)$

$f_n \not\to 0$ λ-a.e.

$\{x : f_n(x) \not\to 0\} = [0,1]$.

With additional hypotheses, something can be salvaged...

$\exists g \in L^1(\mu)$ s.t. $|f_n| \leq g$ μ-a.e. $\forall n \in \mathbb{N}$

(3) + dominating $f_n \Rightarrow$ (4)

To see this...

- $|f_n - f| \leq |g| + |f|$
- $\lim_{n\to\infty} |f_n - f| = 0$ μ-a.e., so

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu \overset{\text{DCT}}{=} \int \lim_{n\to\infty} |f_n - f| \, d\mu = 0$$

We will now work up to showing (4) $\Rightarrow$ (3) "up to a subsequence"

Convergence in Measure

Def: A sequence of measurable functions $f_n : X \to \mathbb{R}$ converges in measure to a measurable function $f : X \to \mathbb{R}$ if, $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Likewise, f_n is Cauchy in measure if, $\forall \varepsilon > 0$,

$$\lim_{n,m \to \infty} \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

Rmk: Convergent in measure implies Cauchy in measure, since...

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

$$\subseteq \{x : |f_n - f(x)| \geq \frac{\varepsilon}{2}\} \cup \{x : |f_m(x) - f(x)| \geq \frac{\varepsilon}{2}\}$$

$$\Downarrow$$

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n - f(x)| \geq \frac{\varepsilon}{2}\}) + \mu(\{x : |f_m(x) - f(x)| \geq \frac{\varepsilon}{2}\})$$

Thus, sending $n, m \to +\infty$ gives the desired result.

Thm: Consider $f_n : X \to \mathbb{R}$ measurable.

(i) If f_n is Cauchy in measure, then $\exists\, f : X \to \mathbb{R}$ measureable s.t.

(a) $f_n \to f$ in measure

(b) $\exists\, f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

(ii) If in addition, $f_n \to g$ in measure, then $f = g$ a.e.

END OF MATERIAL FOR MIDTERM 2