Lecture 13 - Math 201a

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- Midterm 2 - Thursday, Nov 12th

Modes of Convergence

$(X, \mathcal{M}, \mu)$, $f_n, f : X \to \mathbb{R}$ meas

① uniform convergence

$\Downarrow$ $\sup_{x \in X} |f_n(x) - f(x)| \to 0$

② pointwise convergence

$f_n(x) \to f(x) \quad \forall x \in X$

$\Downarrow$

③ pointwise a.e. convergence

$f_n(x) \to f(x) \quad \mu\text{-a.e. } x \in X$

④ L^1 convergence, $\|f_n - f\|_{L^1(\mu)} \to 0$

How are ③ and ④ related?

③ + dominating fn ⇒ ④

showing ④ ⇒ ③ "up to a subsequence"

Convergence in measure

Def: A sequence of measurable functions $f_n : X \to \mathbb{R}$ converges in measure to a measurable function $f : X \to \mathbb{R}$ if, $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Likewise, f_n is Cauchy in measure if, $\forall \varepsilon > 0$,

$$\lim_{n,m \to \infty} \mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

Rmk: Convergent in measure implies Cauchy in measure.

Thm: Consider $f_n : X \to \mathbb{R}$ measurable.

(i) If f_n in Cauchy in measure, then $\exists\, f : X \to \mathbb{R}$ measureable s.t.

(a) $f_n \to f$ in measure

(b) $\exists\, f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

(ii) If in addition, $f_n \to g$ in measure, then $f = g$ μ-a.e.

END OF MATERIAL FOR MIDTERM 2

Pf: First, we must find our guess for f.

Since f_n is Cauchy in measure, $\forall k \in \mathbb{N}$, $\exists$ subsequence $g_k := f_{n_k}$ s.t.

$$\mu\left(\underbrace{\left\{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k}\right\}}_{E_k}\right) \leq \frac{1}{2^k}$$

($g_k = f_{n_k}$, $g_{k+1} = f_{n_{k+1}}$)

By countable subadditivity, $\forall \ell \in \mathbb{N}$,

$$\mu\left(\underbrace{\bigcup_{k=\ell}^{\infty} E_k}_{F_\ell}\right) \leq \sum_{k=\ell}^{\infty} \mu(E_k) \leq \frac{1}{2^{\ell-1}}.$$

(*) Note if $x \notin F_\ell$, then for $i \geq j \geq \ell$,

$$|g_i(x) - g_j(x)| \leq \sum_{k=j}^{i} |g_k(x) - g_{k+1}(x)|$$

$$\leq \sum_{k=j}^{i} \frac{1}{2^k} \leq \frac{1}{2^{j-1}}.$$

$x \notin F_\ell$ for some $\ell \in \mathbb{N}$

$\Updownarrow$

$x \in F_\ell^c$ for some $\ell \in \mathbb{N}$

$\Downarrow$

$x \in \bigcup_{\ell=1}^{\infty} F_\ell^c$

$G := \bigcup_{\ell=1}^{\infty} F_\ell^c \iff x \notin \bigcap_{\ell=1}^{\infty} F_\ell$

!! G^c

Thus $x \notin F_\ell$ for some $\ell \in \mathbb{N}$, then $\{g_i(x)\}_{i=1}^{\infty}$ is Cauchy, so it converges to $\lim_{i\to\infty} g_i(x) \in \mathbb{R}$.

Define $f(x) := \begin{cases} \lim_{i\to\infty} g_i(x) & x \in G \\ 0 & x \in G^c \end{cases}$

To show part (b), that $g_k = f_{n_k} \to f$ μ-a.e., it suffices to show $\mu(G^c) = 0$. This holds since, $\forall \ell \in \mathbb{N}$,

$$\mu(G^c) = \mu\left(\bigcap_{\ell=1}^{\infty} F_\ell\right) \le \mu(F_\ell) \le \frac{1}{2^{\ell-1}}$$

- Sending $\ell \to +\infty$ shows $\mu(G^c) = 0$. This completes the proof of (b).
- Now, to show (a), we show $f_n \to f$ in measure. First, show $g_i \to f$ in measure.

CLAIM: $\{x : |g_\ell(x) - f(x)| > \frac{1}{2^{\ell-1}}\} \subseteq F_\ell$

Pf: $x \notin F_\ell \overset{(*)}{\Rightarrow} \forall i, j \geq \ell, |g_i(x) - g_j(x)| \leq \frac{1}{2^{j-1}}$ ($i \to +\infty \Rightarrow j = \ell$)

$\Rightarrow |f(x) - g_\ell(x)| \leq \frac{1}{2^{\ell-1}}$.

Fix $\varepsilon > 0$. Choose $\ell \in \mathbb{N}$ large enough so $\varepsilon > \frac{1}{2^{\ell-1}}$. Then,

$$\mu(\{x : |g_\ell(x) - f(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |g_\ell(x) - f(x)| > \tfrac{1}{2^{\ell-1}}\})$$

CLAIM

$$\leq \mu(F_\ell) \leq \frac{1}{2^{\ell-1}} \xrightarrow{\ell \to \infty} 0$$

Thus $g_k \to f$ in measure.

$g_k = f_{n_k}$

Finally,

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$$

$$\leq \mu(\{x : |f_n(x) - f_{n_k}(x)| \geq \frac{\varepsilon}{2}\}) + \mu(\{x : |f_{n_k}(x) - f(x)| \geq \frac{\varepsilon}{2}\})$$

$\xrightarrow{n,k \to +\infty} 0,$

since f_n is Cauchy in measure.

This shows $f_n \to f$ in measure.

Finally, to show (ii),
suppose $f_n \to g$ in measure.

$$\mu(\{x : |f(x) - g(x)| \geq \varepsilon\})$$

$$\leq \mu\left(\left\{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2}\right\}\right)$$

$$+ \mu\left(\left\{x : |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}\right)$$

By convergence in measure,
RHS $\xrightarrow{n \to \infty} 0$, $\forall \varepsilon > 0$.

Thus, $\forall \varepsilon > 0$, $\mu(\{x : |f(x) - g(x)| \geq \varepsilon\}) = 0$.

$$\mu(\{x : |f(x) - g(x)| > 0\})$$

$$= \mu\left(\bigcup_{k=1}^{\infty} \left\{x : |f(x) - g(x)| \geq \frac{1}{k}\right\}\right)$$

$$\leq \sum_{k=1}^{\infty} \mu\left(\left\{x : |f(x) - g(x)| \geq \frac{1}{k}\right\}\right)$$

$$= \sum_{k=1}^{\infty} 0 = 0.$$

□

We now apply this to study convergence in $L^1(\mu)$.

Prop:

(a) If f_n is Cauchy in $L^1(\mu)$, then it's Cauchy in measure.

(b) If convergent in $L^1(\mu)$, then convergent in measure.

Pf: First, we show (a) $\Rightarrow$ (b).

If $f_n \to f$ in $L^1(\mu)$, then f_n is Cauchy in $L^1(\mu)$, so by (a), f_n is Cauchy in measure. Prev thm shows convergent in measure.

It remains to show (a).
Fix $\varepsilon > 0$. Define

$$E_{n,m,\varepsilon} := \{x : |f_n(x) - f_m(x)| \geq \varepsilon\}$$

Then

$$\varepsilon \mu(E_{n,m,\varepsilon}) = \int_{E_{n,m,\varepsilon}} \varepsilon \, d\mu$$

$$\leq \int_{E_{n,m,\varepsilon}} |f_n - f_m| \, d\mu \leq \|f_n - f_m\|_{L^1(\mu)}$$

Since RHS $\xrightarrow{n,m \to +\infty}$ 0, $\varepsilon \mu(E_{n,m,\varepsilon}) \to 0$ as $n,m \to \infty$.
Since $\varepsilon > 0$, $\mu(E_{n,m,\varepsilon}) \xrightarrow{n,m \to \infty} 0$,
so f_n is Cauchy in measure. □

Cor: If f_n is Cauchy in $L^1(\mu)$, then $\exists\, f \in L^1(\mu)$ and a subsequence f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

Pf: The fact that $\exists f: X \to \mathbb{R}$ and f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e. follows immediately by combining previous two results.

To see $f \in L^1(\mu)$, use Fatou:

$$\begin{aligned}\int |f| d\mu &= \int \lim_{k\to\infty} |f_{n_k}| d\mu \\ &= \int \liminf_{k\to\infty} |f_{n_k}| d\mu \\ &\le \liminf_{k\to\infty} \int |f_{n_k}| d\mu \\ &\le \sup_n \|f_n\|_{L^1(\mu)} \\ &< +\infty,\end{aligned}$$

since all Cauchy sequences in a metric space are bounded. □

Cor: $L^1(\mu)$ is a Banach space, that is, a complete normed vector space.

Pf: Fix f_n Cauchy in $L^1(\mu)$.

By previous Cor, $\exists f \in L^1(\mu)$ and subsequence f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

$$\int |f_{n_k} - f| \, d\mu = \int \lim_{j\to\infty} |f_{n_k} - f_{n_j}| \, d\mu$$

$$= \int \liminf_{j\to\infty} |f_{n_k} - f_{n_j}| \, d\mu$$

$$\leq \liminf_{j\to\infty} \int |f_{n_k} - f_{n_j}| \, d\mu$$

$$\xrightarrow{k\to+\infty} 0$$

so $f_{n_k} \to f$ in $L^1(\mu)$.

Lastly, a general fact of metric spaces ensures that, for any Cauchy sequence, if there is a convergent subsequence, the whole sequence must converge to same limit!

MODES OF CONVERGENCE

Next: product measures