Lecture 14 - Math 201a

©Katy Craig, 2026

HW7 due Sunday, November 22nd

Prop:

(a) If f_n is Cauchy in $L^1(\mu)$, then it's Cauchy in measure.

(b) If convergent in $L^1(\mu)$, then convergent in measure.

Cor: If f_n is Cauchy in $L^1(\mu)$, then $\exists\ f \in L^1(\mu)$ and a subsequence f_{n_k} s.t. $f_{n_k} \to f$ μ-a.e.

Cor: $L^1(\mu)$ is a Banach space, that is, a complete normed vector space.

Summary of Modes of Conv:

$f_n \to f$ in $L^1(\mu)$

$\Downarrow$

$f_n \to f$ in measure

$\Downarrow$

$\exists f_{n_k}$ s.t. $f_{n_k} \to f$ μ-a.e.

~~$\frac{1}{n}\mathbb{1}_{[0,n]}$~~

$\Uparrow$ if $\mu(X)<+\infty$, mid2, Q3 shows a.e. conv implies conv in measure

useful on HW7

On HW7, Q2, will prove Egoroff's Theorem ← Know statement

"If $\mu(X)<+\infty$, a.e. conv implies uniform conv, off a small set"

Product Measures

first: product σ-algebras

X is a nonempty set

Recall:

- Given $\mathcal{E} \subseteq 2^X$, there exists a smallest σ-algebra containing $\mathcal{E}$, denoted $\mathcal{M}(\mathcal{E})$.
- $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F}) \Rightarrow \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$
- $\mathcal{E} \subseteq \mathcal{F} \Rightarrow \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$

Let

- $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ a countable collection of measureable spaces
- $X = \prod_{\alpha \in A} X_\alpha$
- $\pi_\alpha : X \to X_\alpha$ denotes the projections

Def: The product σ-algebra is

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha := \mathcal{M}\left(\left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha \right\} \right)$$

"rectangles"

$$\mathcal{E}_X : \{(\mathbb{R}, \mathcal{B}_{\mathbb{R}})\}_{i=1}^{d}$$

$$\bigotimes_{i=1}^{d} \mathcal{B}_{\mathbb{R}} = \mathcal{M}\left(\{ E_1 \times E_2 \times \dots \times E_d : E_i \in \mathcal{B}_{\mathbb{R}} \} \right)$$

Special case: $E_i = (a_i, b_i)$

$$E_1 \times E_2 \times \dots \times E_d = (a_1, b_1) \times \dots \times (a_d, b_d)$$

Goal: $\bigotimes_{i=1}^{d} \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^d}$

Prop: Given $\mathcal{E}_\alpha \subseteq 2^{X_\alpha}$ s.t. $X_\alpha \in \mathcal{E}_\alpha$, then for $\mathcal{M}_\alpha := \mathcal{M}(\mathcal{E}_\alpha)$,

$$\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \mathcal{M}\left(\left\{ \prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha \right\} \right)$$

Pf: HW7, Q8

Ex: $\mathcal{E}_L = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}\}$

Thm: Given metric spaces $X_1, \ldots, X_d$ and $X = \prod_{i=1}^{d} X_i$ is endowed with the metric

$$d_{max}((x_1, \ldots, x_d), (y_1, \ldots, y_d)) = \max_{i=1,\ldots,d} d_i(x_i, y_i)$$

we have $\bigotimes_{i=1}^{d} \mathcal{B}_{X_i} \subseteq \mathcal{B}_{\prod_{i=1}^{d} X_i}$.

Furthermore, if $\{X_i\}_{i=1}^{d}$ are separable, then

$$\bigotimes_{i=1}^{d} \mathcal{B}_{X_i} = \mathcal{B}_{\prod_{i=1}^{d} X_i}.$$

Rmk: d_{max} is convenient because $B_r(x_1,\dots,x_d) = \prod_{i=1}^{d} B_r(x_i)$. However, since the defn of B_X only depends on the topology of X, the theorem continues to hold when X is endowed with any equivalent metric.

$$c\,d_2(x,y) \le d_1(x,y) \le C\,d_2(x,y)$$

Recall:

A a metric space is separable in case it has a countable dense subset

B $\{X_i\}_{i=1}^{d}$ separable $\Rightarrow X$ separable

C In a separable metric space, every open set can be written as a countable union of (open) balls.

D $\{E_i\}_{i=1}^d$ open $\Rightarrow \prod_{i=1}^d E_i$ open

Pf: By Prop,

$$\bigotimes_{i=1}^d \mathcal{B}_{X_i} = \mathcal{M}\left(\left\{\prod_{i=1}^d E_i : E_i \subseteq X_i, E_i \text{ open}\right\}\right)$$

$$\subseteq \mathcal{M}(\{U : U \subseteq X \text{ open}\})$$

$$= \mathcal{B}_X$$

We now prove the opposite containment, under the hypothesis that X_i are separable

It suffices to show

$$\{U : U \subseteq X \text{ open}\} \subseteq \bigotimes_{i=1}^d \mathcal{B}_{X_i}$$

By C it suffices to show

for any (open) ball B,

$$B \in \bigotimes_{i=1}^{d} \mathcal{B}_{X_i}.$$

Since X is endowed w/ d_{max},
$B = \prod_{i=1}^{d} B_i$, $B_i \subseteq X_i$ (open) balls.

$$\left\{ \prod_{i=1}^{d} B_i : B_i \text{ open} \right\} \subseteq \left\{ \prod_{i=1}^{d} E_i, E_i \text{ open} \right\}$$

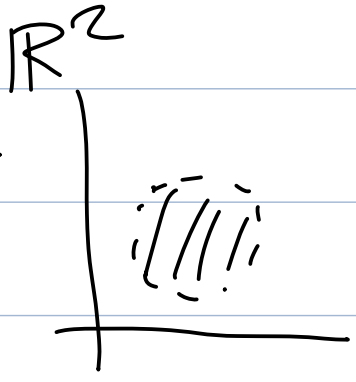
$$\cap$$

$$\mathcal{M}\left(\left\{ \prod_{i=1}^{d} E_i : E_i \subseteq X_i, E_i \text{ open} \right\} \right)$$

$$\| $$

$$\bigotimes_{i=1}^{d} \mathcal{B}_{X_i}$$ □

Ex:



Now: product measures

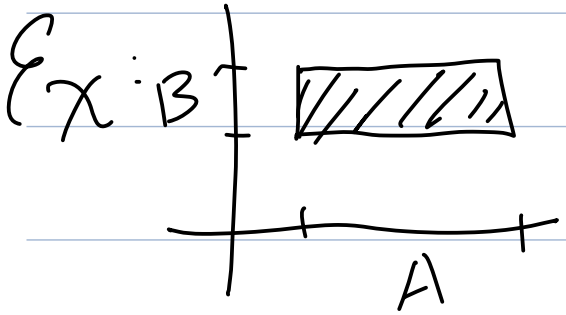
Measure spaces $(X, \mathcal{M}, \mu)$
$(Y, \mathcal{N}, \nu)$

Rectangles $A \times B$, $A \in \mathcal{M}, B \in \mathcal{N}$

$\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\})$

Goal: Prove existence of a unique measure ω on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ with the property that

$\omega(A \times B) = \mu(A)\nu(B), \forall A \in \mathcal{M}$
$B \in \mathcal{N}$



We will denote the product measure by $\mu \otimes \nu := \omega$.

A big theorem we will use to accomplish this goal is the Monotone Class Theorem.

Def: $\mathcal{C} \subseteq 2^X$ is a monotone class if it's a nonempty collection s.t.

"closed under countable increasing unions"

(i) $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}, E_1 \subseteq E_2 \subseteq \ldots \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$

"closed under countable decreasing intersections"

(ii) $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{C}, E_1 \supseteq E_2 \supseteq \ldots \Rightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{C}$

Ex: Any σ-algebra is a monotone class.

Ex: $X = \{1,2,3\}$
$\mathcal{C} = \{\{\{1\}, \{1,2\}, \{1,2,3\}\}$ is a monotone class, but not σ-algebra.

Prop: Given $\mathcal{E} \subseteq 2^X$ nonempty there exists a smallest monotone class containing $\mathcal{E}$, denoted $\mathcal{C}(\mathcal{E})$.

Pf: Homework.

Thm: Given an algebra $\mathcal{A} \subseteq 2^X$,

$$\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$$

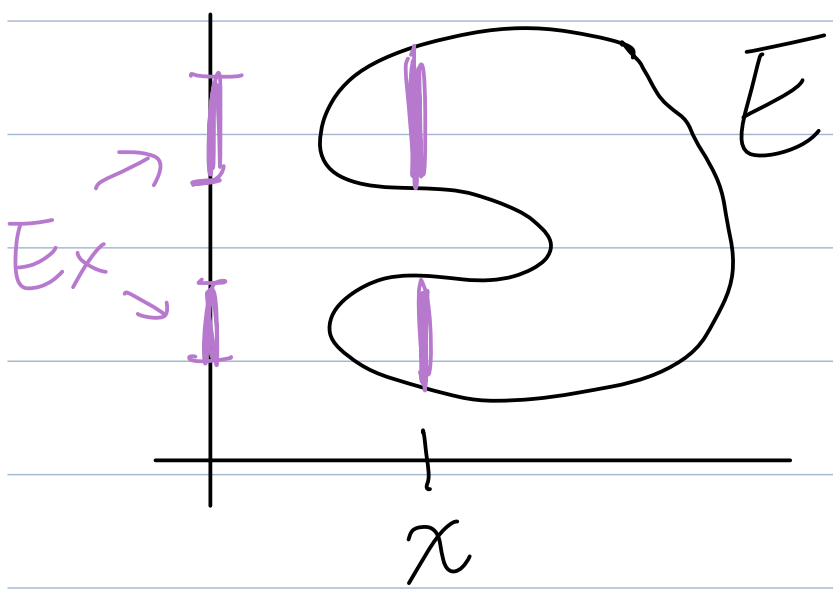
Will prove later.

A little more terminology for existence and uniqueness of product measures...

Def: For any $E \in \mathcal{M} \otimes \mathcal{N}$,

the x-section $E_x = \{y : (x,y) \in E\}$

y-section $E^y = \{x : (x,y) \in E\}$



Basic properties of sections:

(A) If $E = A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$,

$$E_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}, \quad E^y = \begin{cases} A & y \in B \\ \emptyset & y \notin B \end{cases}$$

(B) Given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M} \otimes \mathcal{N}$

$$\left(\bigcup_{i=1}^{\infty} E_i\right)_x = \{y : (x,y) \in \bigcup_{i=1}^{\infty} E_i\}$$

$$= \bigcup_{i=1}^{\infty} \{y : (x,y) \in E_i\}$$

$$= \bigcup_{i=1}^{\infty} E_i^x$$

(C) Given $E \in \mathcal{M} \otimes \mathcal{N}$

$$(E^c)_x = (E_x)^c$$

(D) Prop: Given any $E \in \mathcal{M} \otimes \mathcal{N}$, $E_x \in \mathcal{N}$ (*) and $E^y \in \mathcal{M}$ (**), $\forall x, y$.

Pf: Let $\mathcal{E} = \{E \in \mathcal{M} \times \mathcal{N} : (*), (**)$ hold $\forall x, y\}$

By (A), $\mathcal{E}$ contains all rectangles.

By (B) and (C), $\mathcal{E}$ is a σ-algebra.

Thus $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{E}$. $\square$