

Lecture 1 - Math 201a

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Key motivating question of measure theory:

What is the "total size" of an arbitrary set $E \subseteq \mathbb{R}^d$?

More precisely, our first goal will be...

Can we define a function

$$\mu: 2^{\mathbb{R}^d} \rightarrow [0, +\infty]$$

so that...

Given a set X , 2^X denote its power set

① μ given the "right size" to "simple sets", e.g. $d=1$
 $\mu([a,b]) = b-a$ for $a \leq b$

$$E_1, E_2, \dots, E_n \subseteq \mathbb{R}^d$$

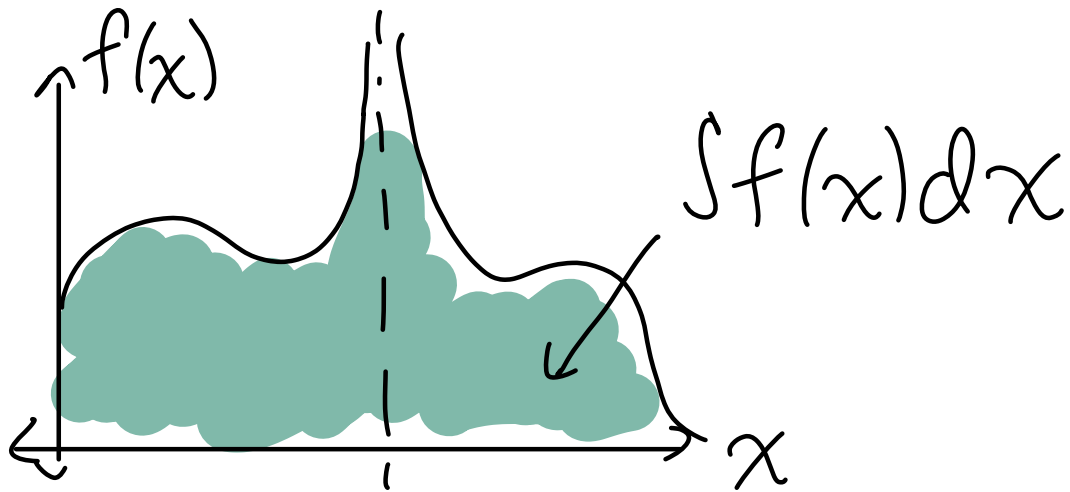
② μ is finitely additive
If $\{E_i\}_{i=1}^n \subseteq \mathcal{Z}^{\mathbb{R}^d}$ are disjoint
then $\mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$.

② μ is countably additive
If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{Z}^{\mathbb{R}^d}$ are disjoint
then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

③ μ is translation invariant
For all $E \subseteq \mathbb{R}^d$, $c \in \mathbb{R}^d$,
 $\mu(E+c) = \mu(E)$

$$\{x+c : x \in E\}$$

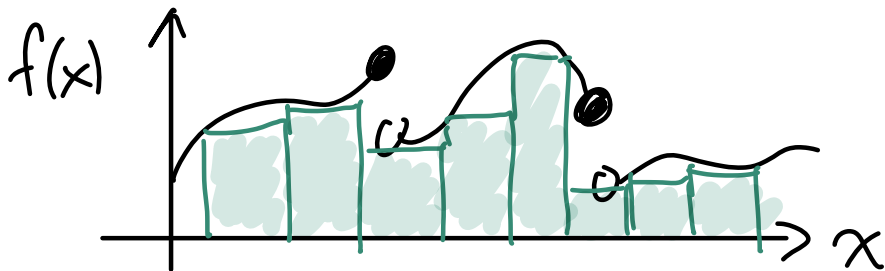
An important set we want to measure is the "area" under a function:



Pre-measure theory:

Riemann Integral (mid 1800s)

rigorous defn of integral of,
e.g., piecewise continuous
functions, in terms of
upper/lower sums



PROS Good enough for "ordinary" fn

CONS Not good enough for "unusual" fns, e.g.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

or taking limits

For example, given $f_n: [a, b] \rightarrow \mathbb{R}$
s.t. $\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \forall x \in [a, b]$

When can we conclude

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx?$$

Measure theory provides a more powerful theory of integration than the Riemann integral.

PROS

- can integrate a much larger class of functions
- better suited to taking limits
- better suited to probability, functional analysis, PDE ✓

CONS

- None :-

Back to our goal...

Thm (Vitali): There does not exist $\mu: 2^{\mathbb{R}} \rightarrow [0, +\infty]$ that satisfies $\mu([a, b]) = b - a$ for $a \leq b$, is countably additive, and translation invariant.

We will use the following lemma:

Lemma (Monotonicity): Given a set X and $\mu: 2^X \rightarrow [0, +\infty]$ finitely additive, then $\forall A, B \subseteq X$,

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

Pf:

$$\begin{aligned}\mu(B) &= \mu(A \cup (B \setminus A)) \\ &= \mu(A) + \mu(B \setminus A) \\ &\geq \mu(A)\end{aligned}$$

$B \cap A^c$



□

Pf of Thm:

Assume, for the sake of contradiction, that there exists such a μ .

Define an equivalence relation on $\mathbb{R}$:

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

$$[x] = \{y \in \mathbb{R} : y \sim x\}$$

Claim 1: Every equivalence class contains an element in $[0, 1]$. (HW1, 6a)

For each equivalence class, choose an element in $[0, 1]$ belonging to that class, and let A be the set of elements that you choose

Warning: Axiom of Choice
Bogachov 1.12(x) ✓

$$\text{Let } B = \bigcup_{q \in \mathbb{Q} \cap [0, 1]} A + q$$

Claim 2: This is a disjoint union
(HW1, 6b)

Claim 3: $[0, 1] \subseteq B \subseteq [-1, 2]$
(i) (ii)

(ii): Since $A \subseteq [0, 1]$, $q \in [-1, 1]$,
 $A + q \subseteq [-1, 2] \forall q \in \mathbb{Q} \cap [-1, 1]$.
 $\Rightarrow B \subseteq [-1, 2]$.

(i): Fix $x \in [0, 1]$. Then
 $x \in [a]$ for some $a \in A$.
 $\cap$
 $[0, 1]$

Then $x - a = q$ for $q \in \mathbb{Q} \cap [-1, 1]$

Therefore $x \in B$.

We can now obtain a contradiction.

By monotonicity lemma,

$$1 = \mu([0, 1]) \leq \mu(B) \leq \mu([-1, 2]) = 3$$

By countable additivity and translation invariance,

$$\begin{aligned}\mu(B) &= \mu\left(\bigcup_{q \in \mathbb{Q} \cap [-1, 1]} A + q\right) \\ &= \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mu(A + q) \\ &= \sum_{q \in \mathbb{Q} \cap [-1, 1]} \mu(A)\end{aligned}$$

If $\mu(A) = 0$, then $\mu(B) = 0 \neq 1$.

If $\mu(A) > 0$, then $\mu(B) = +\infty \neq 3$.

Thus, no such μ can exist. $\square$

Which criterion should we weaken to get existence?

Not size of intervals...

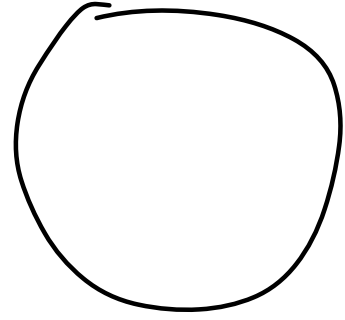
Not translation invariance...

Maybe weaken countable additivity to finite additivity?

Still problems for $d \geq 3$.

Banach-Tarski Paradox (1924)

$$U = \bigcup_{i=1}^n E_i$$



$$V = \bigcup_{i=1}^n F_i$$

F_i is a rotation/translation
of E_i

Two good choices:

- don't require μ to be defined on all $2^{\mathbb{R}^d}$
- still define μ on all $2^{\mathbb{R}^d}$ but replace countable additivity with countable subadditivity

$\downarrow$

$$\forall \{E_i\}_{i=1}^{\infty} \text{ disjoint}$$
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

outer
measures

We will see that...

- given a measure on "nice enough" subsets of $\mathbb{R}^d$, it extends to an outer measure on $2^{\mathbb{R}^d}$.

- given an outer measure, we can find a collection "nice enough" on which it is a measure

Measures

What subcollection of $2^{\mathbb{R}^d}$ should we restrict to?

Let X be a nonempty set.

Def: $\mathcal{A} \subseteq 2^X$ is an algebra of subsets of X if it is nonempty and

$$(1) \{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathcal{A}$$

$$(2) E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$$

- ① $\Rightarrow$ "closed under finite unions"
② $\Rightarrow$ "closed under complements"

Lemma: If $\mathcal{A}$ is an algebra,
then

① $\emptyset \in \mathcal{A}, X \in \mathcal{A}$

② $\{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcap_{i=1}^n E_i \in \mathcal{A}$
 $\uparrow$ "closed under finite intersections"

Pf: HW 2, Q2

$\mathcal{E}_X$:

(i) $\mathcal{A} = 2^X$

(ii) $\mathcal{A} = \{\emptyset, X\}$

(iii) $\mathcal{A}$ is the collection of
all finite and cofinite
subsets of X .

