

Lecture 2 - Math 201a

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Recall:

Thm (Vitali): There does not exist $\mu: 2^{\mathbb{R}} \rightarrow [0, +\infty]$ that satisfies $\mu([a, b]) = b - a$ for $a \leq b$, is countably additive, and translation invariant.

Def: μ is countably additive, $n = +\infty$
finitely additive if
for $\{E_i\}_{i=1}^n \subseteq 2^X$ disjoint,

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$$

Lemma (Monotonicity): Given
a set X and $\mu: 2^X \rightarrow [0, +\infty]$
finitely additive, then
 $\forall A, B \subseteq X,$

$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B).$$

Two good choices:

- don't require μ to be defined on all $2^{\mathbb{R}^d}$
- still define μ on all $2^{\mathbb{R}^d}$ but replace countable additivity with countable subadditivity

$\downarrow$

$\forall \{E_i\}_{i=1}^{\infty} \text{ disjoint}$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

Measures

Let X be a nonempty set.

Def: $\mathcal{A} \subseteq 2^X$ is an algebra of subsets of X if it is nonempty and

$$(1) \{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathcal{A}$$

$$(2) E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$$

Lemma: If $\mathcal{A}$ is an algebra, then

$$(1) \emptyset \in \mathcal{A} \quad X \in \mathcal{A}$$

$$(2) \{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcap_{i=1}^n E_i \in \mathcal{A}$$

Def: $\mathcal{A} \subseteq 2^X$ is a σ -algebra of subsets of X if it is an algebra and it's closed under countable unions.

$$\downarrow$$
$$\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$$

Rmk: A σ -algebra is also closed under countable intersections.

Lemma: An algebra $\mathcal{A}$ that is closed under countable disjoint unions is a σ -algebra.

↓

$$\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}, E_i \text{ disjoint} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$$

Pf: HW2, Q1

Ex:

(i) $\mathcal{A} = 2^X$

(ii) $\mathcal{A} = \{\emptyset, X\}$

(iii) $\mathcal{A} = \{\text{all finite and cofinite subsets of } X\}$

} σ -algebras

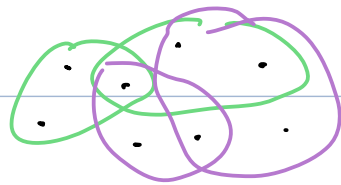
not a σ -algebra
↓ if $|X| = +\infty$.

Def: Given a nonempty set X and a σ -algebra $\mathcal{M} \subseteq 2^X$, we call $(X, \mathcal{M})$ a measurable space and $E \in \mathcal{M}$ a measurable set.

Prop: Given $E \subseteq 2^X$, there exists a smallest σ -algebra containing E , which we denote $\mathcal{M}(E)$, and refer to as the σ -algebra generated by E .

that is, all other σ -algebras $\tilde{\mathcal{F}}$ containing E satisfy $\mathcal{M}(E) \subseteq \tilde{\mathcal{F}}$.

Pf:



Claim Given any nonempty collection $\mathcal{C}$ of σ -algebras on X , then

$$\bigcap \mathcal{C} = \{E \subseteq X : E \in \mathcal{A} \forall \mathcal{A} \in \mathcal{C}\}$$

is a σ -algebra.

$\uparrow$ equiv...

$$\bigcap_{\alpha \in I} \mathcal{A}_\alpha, \mathcal{C} = \{\mathcal{A}_\alpha\}_{\alpha \in I}$$

Pf of Claim: HW2, Q2

Let $\mathcal{C}$ be collection of all σ -algebras containing $\mathcal{E}$.

Since $2^X \in \mathcal{C}$, $\mathcal{C}$ is nonempty,

so, by claim,

$\bigcap \mathcal{C}$ is a σ -algebra.

Since $E \subseteq \mathcal{A}$ for all in $\mathcal{C}$,
 $E \subseteq \bigcap \mathcal{C}$.

Finally, if $\mathcal{F}$ is a σ -algebra
containing E , $\mathcal{F} \in \mathcal{C}$,
so $\bigcap \mathcal{C} \subseteq \mathcal{F}$.

Therefore $\bigcap \mathcal{C} = \mathcal{M}(E)$. $\square$

Recall: A topology $\mathcal{T}$ on X
is a collection of subsets
of X that

- contains $\emptyset$ and X
- is closed under arbitrary
unions and finite
intersections

The elements of $\mathcal{T}$, $T \in \mathcal{T}$,
are called open sets.

Let (X, τ) be a topological space.

Def: The Borel σ -algebra of X , denote $\mathcal{B}(X)$, is the σ -algebra generated by τ . Its elements are called Borel sets.

Prop: The Borel σ -algebra of $\mathbb{R}$, denote $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following:

- (i) open intervals, $\mathcal{E}_1 := \{(a, b) : a < b\}$
- (ii) closed intervals, $\mathcal{E}_2 := \{[a, b] : a \leq b\}$
- (iii) half-open intervals, $\mathcal{E}_3 := \{[a, b) : a < b\}$
- (iv) open rays, $\mathcal{E}_4 := \{(a, +\infty) : a \in \mathbb{R}\}$

Pf: HW2, Q4

Measures

Def: Given a measurable space $(X, \mathcal{M})$, a measure is a function $\mu: \mathcal{M} \rightarrow [0, +\infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ disjoint,
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a measure space.

Ex: (Dirac mass / Dirac measure)

$$(\mathcal{X}, \mathcal{M}) = (\mathcal{X}, 2^{\mathcal{X}})$$

For $x_0 \in \mathcal{X}$, define

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$$

Often denote $\mu = \delta_{x_0}$.

Ex: (counting measure)

$$(\mathcal{X}, \mathcal{M}) = (\mathcal{X}, 2^{\mathcal{X}})$$

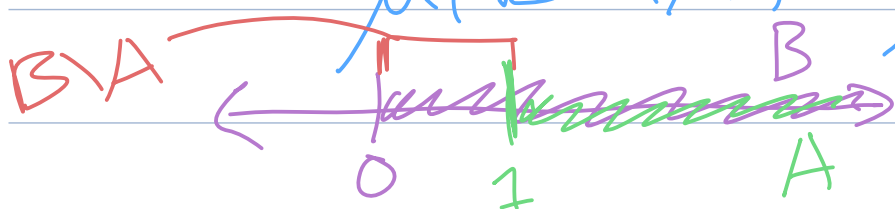
$$\mu(A) = |A| = \# \text{ of elements in } A.$$

Thm: For any measure space $(\mathcal{X}, \mathcal{M}, \mu)$ and $A, B \in \mathcal{M}$, $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$,

$$(i) A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

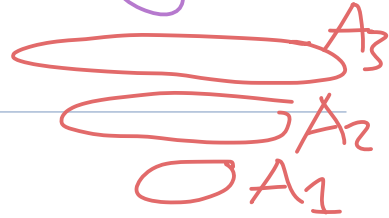
$$(ii) A \subseteq B, \mu(A) < +\infty$$

$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$$



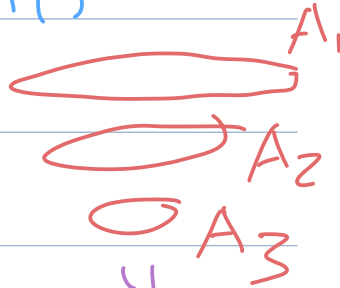
(iii) $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$
 not necessarily disjoint
 "countable subadditivity"

(iv) $A_i \subseteq A_{i+1}, \forall i \in \mathbb{N}$
 $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$



"continuity from below"

(v) $A_i \supseteq A_{i+1}, \forall i \in \mathbb{N}, \mu(A_1) < +\infty$
 $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$



"continuity from above"

Pf:

(i) See Lec 1.

(ii) $\mu(B) = \mu((B \setminus A) \cup A)$
 $= \mu(B \setminus A) + \mu(A)$
 $\mu(B) - \mu(A) = \mu(B \setminus A)$

$\downarrow \mu(A) < +\infty$

(iii) Define $B_1 := A_1$, $B_2 := A_2 \setminus A_1, \dots$
 $\dots B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$

Then B_n are disjoint and
 $\bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} A_i$.

By countable additivity,
 $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$

$$= \sum_{n=1}^{\infty} \mu(B_n) \quad \left. \begin{array}{l} B_n \subseteq A_n \\ (i) \end{array} \right\}$$

$$\leq \sum_{n=1}^{\infty} \mu(A_n)$$

(iv)

Define $B_1 = A_1$
 $B_{n+1} = A_{n+1} \setminus A_n$



Then $\{B_n\}_{n=1}^{\infty}$ are disjoint

and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} A_i$.

Then, $\forall n \in \mathbb{N}$, $\mu(A_n) = \mu\left(\bigcup_{k=1}^n B_k\right)$
 $= \sum_{k=1}^n \mu(B_k),$

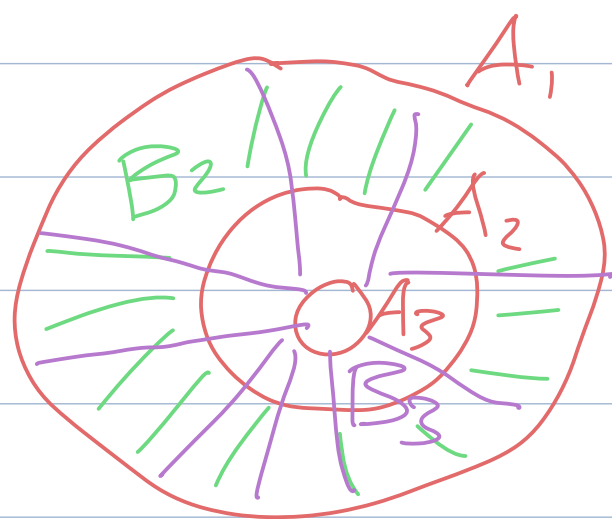
so $\lim_{n \rightarrow \infty} \mu(A_n) = \sum_{k=1}^{\infty} \mu(B_k)$

$= \mu\left(\bigcup_{k=1}^{\infty} B_k\right)$

$= \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$

(v)

Define $B_i = A_1 \setminus A_i$.



By construction,

$B_i \subseteq B_{i+1} \quad \forall i \in \mathbb{N}.$

Furthermore

$$\begin{aligned}\bigcup_{i=1}^{\infty} B_i &= \bigcup_{i=1}^{\infty} (A_1 \setminus A_i) = \bigcup_{i=1}^{\infty} A_1 \cap A_i^c \\ &= A_1 \cap \left(\bigcup_{i=1}^{\infty} A_i \right)^c = A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i \right)^c \\ &= A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i \right)\end{aligned}$$

Therefore

$$\mu(A_1) = \underbrace{\mu\left(A_1 \setminus \left(\bigcap_{i=1}^{\infty} A_i\right)\right)}_{\underbrace{\bigcup_{i=1}^{\infty} B_i}} + \underbrace{\mu\left(A_1 \cap \left(\bigcap_{i=1}^{\infty} A_i\right)\right)}_{\bigcap_{i=1}^{\infty} A_i}$$

(iv) ↓

$$= \lim_{i \rightarrow \infty} \mu(B_i) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Since $\mu(A_1) < +\infty$
 $\mu(B_i) = \mu(A_1 \setminus A_i)$
 (iii) $\downarrow = \mu(A_1) - \mu(A_i)$

$$\dots = \lim_{i \rightarrow \infty} \mu(A_1) - \mu(A_i) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Since $\mu(A_1) < +\infty$, subtracting from both sides...

$$0 = -\lim_{i \rightarrow \infty} \mu(A_i) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

$\mu(A_1) < +\infty$
 and $A_i \supseteq A_{i+1}$
 ensures finite

$$\lim_{i \rightarrow \infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \quad \square$$