

Lecture 3 - Math 201a

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Recall:

Let X be a nonempty set.

Def: $\mathcal{A} \subseteq 2^X$ is an algebra of subsets of X if it is nonempty and

- (1) $\{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcup_{i=1}^n E_i \in \mathcal{A}$
- (2) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$

Lemma: If $\mathcal{A}$ is an algebra, then

- (1) $\emptyset \in \mathcal{A}$ $X \in \mathcal{A}$
- (2) $\{E_i\}_{i=1}^n \subseteq \mathcal{A} \Rightarrow \bigcap_{i=1}^n E_i \in \mathcal{A}$

Def: $\mathcal{A} \subseteq 2^X$ is a σ -algebra of subsets of X if it is an algebra and it's closed under countable unions.

Lemma: An algebra $\mathcal{A}$ that is closed under countable disjoint unions is a σ -algebra.

Ex:

- (i) $\mathcal{A} = 2^X$
 - (ii) $\mathcal{A} = \{\emptyset, X\}$
 - (iii) $\mathcal{A} = \{\text{all finite and cofinite subsets of } X\}$
- } σ -algebras
- not a σ -algebra
↓ if $|X| = +\infty$.

Def: Given a nonempty set X and a σ -algebra $\mathcal{M} \subseteq 2^X$, we call $(X, \mathcal{M})$ a measurable space and $E \in \mathcal{M}$ a measurable set.

Prop: Given $\mathcal{E} \subseteq 2^X$, there exists a smallest σ -algebra containing $\mathcal{E}$, which we denote $\mathcal{M}(\mathcal{E})$, and refer to as the σ -algebra generated by $\mathcal{E}$.

Let (X, τ) be a topological space.

Def: The Borel σ -algebra of X , denote $\mathcal{B}(X)$, is the σ -algebra generated by τ . Its elements are called Borel sets.

Prop: The Borel σ -algebra of $\mathbb{R}$, denote $\mathcal{B}_{\mathbb{R}}$, is generated by each of the following:

- (i) open intervals, $\mathcal{E}_1 := \{(a, b) : a < b\}$
- (ii) closed intervals, $\mathcal{E}_2 := \{[a, b] : a \leq b\}$
- (iii) half-open intervals, $\mathcal{E}_3 := \{[a, b) : a < b\}$
- (iv) open rays, $\mathcal{E}_4 := \{(a, +\infty) : a \in \mathbb{R}\}$

Measures

Def: Given a measurable space $(X, \mathcal{M})$, a measure is a function $\mu: \mathcal{M} \rightarrow [0, +\infty]$ s.t.

- (i) $\mu(\emptyset) = 0$
- (ii) given $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ disjoint,
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

We call $(X, \mathcal{M}, \mu)$ a measure space.

Ex: (Dirac mass / Dirac measure)

$$(\mathcal{X}, \mathcal{M}) = (\mathcal{X}, 2^{\mathcal{X}})$$

For $x_0 \in \mathcal{X}$, define

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$$

Often denote $\mu = \delta_{x_0}$.

Ex: (counting measure)

$$(\mathcal{X}, \mathcal{M}) = (\mathcal{X}, 2^{\mathcal{X}})$$

$$\mu(A) = |A| = \# \text{ of elements in } A.$$

Thm: For any measure space $(\mathcal{X}, \mathcal{M}, \mu)$ and $A, B \in \mathcal{M}$, $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$,

$$(i) A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

$$(ii) A \subseteq B, \mu(A) < +\infty$$

$$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$$

$$(iii) \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

"countable subadditivity"

$$(iv) A_i \subseteq A_{i+1} \quad \forall i \in \mathbb{N},$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i)$$

"continuity from below"

$$(v) A_i \supseteq A_{i+1} \quad \forall i \in \mathbb{N}, \mu(A_1) < +\infty$$

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i)$$

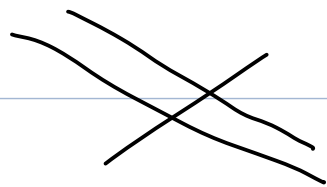
"continuity from above"

Ex: $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, μ is counting measure

$$A_k := \{n \in \mathbb{N} : n \geq k\}$$

$$\mu(A_i) = +\infty$$

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(\emptyset) = 0$$



$$\lim_{i \rightarrow \infty} \mu(A_i) = \lim_{i \rightarrow \infty} (+\infty) = +\infty$$

Measure Terminology

$(X, \mathcal{M}, \mu)$

- μ is a finite measure if $\mu(X) < +\infty$.

- μ is a σ -finite measure if $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty \forall i$.

- $E \subseteq X$ is a null set if $E \in \mathcal{M}$ and $\mu(E) = 0$.

- We say a property holds almost everywhere if the set of points on which it fails is a null set.



Ex: Suppose $f: X \rightarrow \mathbb{R}$.

Then $f = 0$ a.e. if

$\{x : f(x) \neq 0\}$ is a null set.

Return to our original goal of defining a measure that respects our usual notion of "size" in $\mathbb{R}^d$.

To prove such a measure exists...

Outer Measure

Def: An outer measure on X is a function $\mu^*: 2^X \rightarrow [0, +\infty]$ such that

- (i) $\mu^*(\emptyset) = 0$
- (ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
- (iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Rmk:

(ii) + (iii) $\Leftrightarrow$ If $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

Ex: (Lebesgue outer measure)

Define $\mu^*: \mathbb{R} \rightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} b_i - a_i : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

"size of covering
 $\{(a_i, b_i]\}_{i=1}^{\infty}$ "

$a_i \leq b_i$



We will show:

- μ^* is an outer measure
- $\mu^*((a, b]) = b - a, \quad a \leq b$
- μ^* is translation invariant
- ... μ^* is not countably additive:

But it will be countably additive when we restrict it to the "right" σ -algebra.

[Q:] Given an outer measure μ^* , what is a σ -algebra on which it is countably additive?

[A:] Carathéodory σ -algebra.

Sadly, not the largest σ -algebra on which countably additive...

Def: Given an outer measure μ^* on X , we say $A \subseteq X$ is μ^* -measurable if

(i)

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subseteq X$$

"A breaks apart all other sets nicely"

$$\mathcal{M}_{\mu^*} = \{A \subseteq X : A \text{ } \mu^*\text{-measurable}\}$$

Carathéodory σ -algebra.

Rmk: For any $A, E \subseteq X$, we always have " $\leq$ " in (i), so to show A is μ^* -meas, it suffices to show " $\geq$ ".

Thm (Carathéodory): Given an outer measure μ^* ,
(i) $\mathcal{M}_{\mu^*}$ is a σ -algebra
(ii) μ^* is a measure on $\mathcal{M}_{\mu^*}$.

We begin with some preliminary results...

Prop: For any outer measure μ^* , if $\mu^*(B) = 0$, then $B \in \mathcal{M}_{\mu^*}$.

Pl: Fix $E \subseteq X$ arbitrary.

It suffices to show " $\geq$ ".
monotonicity

$$\mu^*(E) \geq \mu^*(E \cap B^c) = \underbrace{\mu^*(E \cap B)}_{=0 \text{ by monoton.}} + \mu^*(E \cap B^c)$$

Prop:

(i) $\mathcal{M}_{\mu^*}$ is algebra

(ii) $\mu^*|_{\mathcal{M}_{\mu^*}}$ is finitely additive
 $\uparrow E=X$

(iii) Given $\{B_i\}_{i=1}^n \subseteq \mathcal{M}_{\mu^*}$ disjoint,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^n B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

$$\forall E \subseteq X.$$

Pf:

Since $\mu^*(\emptyset) = 0$, by prev prop, $\emptyset \in \mathcal{M}_{\mu^*}$; so $\mathcal{M}_{\mu^*}$ is nonempty.

Further $\mathcal{M}_{\mu^*}$ is closed under complements by

Symmetry of (i) in A and A^c .

To see $\mathcal{M}_{\mu^*}$ is closed under finite unions, it suffices to show

$$A, B \in \mathcal{M}_{\mu^*} \Rightarrow A \cup B \in \mathcal{M}_{\mu^*}.$$

Suppose $A, B \in \mathcal{M}_{\mu^*}$. Fix $E \subseteq X$.

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\quad \underbrace{E \cap (A \cup B)}_{\text{purple}} \quad \underbrace{(A \cup B)^c}_{\text{green}} \\ &\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B)) + \mu^*(E \cap (A \cup B)^c) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \end{aligned}$$

Thus, $A \cup B \in \mathcal{M}_{\mu^*}$.

Thus μ^* is an algebra.

Now, we prove (iii).

Fix $E \in \mathcal{X}$, $\{B_i\}_{i=1}^n \in \mathcal{A}_{\mu^*}$ disjoint.

We proceed by induction.

The base case $n=1$ clearly holds $\therefore$.

Suppose the result holds for $n-1$.

$$\begin{aligned} & \mu^*(E \cap (\bigcup_{i=1}^n B_i)) \\ &= \mu^*(E \cap (\bigcup_{i=1}^n B_i) \cap B_n) + \mu^*(E \cap (\bigcup_{i=1}^n B_i) \cap B_n^c) \\ &= \mu^*(E \cap B_n) + \mu^*(E \cap (\bigcup_{i=1}^{n-1} B_i)) \quad \text{inductive} \\ &= \sum_{i=1}^n \mu^*(E \cap B_i). \quad \text{h.p.} \end{aligned}$$

This shows (iii)

Property (ii) then follows
by taking $E = X$.

Prop:

Fix $E \subseteq X$, $\{B_i\}_{i=1}^{\infty} \in \mathcal{M}_{\mu^*}$ disjoint.

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*(E \cap (\bigcup_{i=1}^{\infty} B_i)^c).$$