Lecture 4 - Math 201a

©Katy Craig, 2026

Remark on HW2, Q13

Recall:

Measure Terminology

$(X, \mathcal{M}, \mu)$

- μ is a finite measure if $\mu(X) < +\infty$.

- μ is a σ-finite measure if $\exists \{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$ s.t. $\bigcup_{i=1}^{\infty} E_i = X$ and $\mu(E_i) < +\infty \ \forall i$.

- $E \subseteq X$ is a null set if $E \in \mathcal{M}$ and $\mu(E) = 0$.

- We say a property holds almost everywhere if the set of points on which it fails is a null set.

Outer Measure

Def: An outer measure on X is a function $\mu^*: 2^X \to [0, +\infty]$ such that

(i) $\mu^*(\emptyset) = 0$

(ii) $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Rmk:

(ii)+(iii) $\Leftrightarrow$ If $E \subseteq \bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

Ex: (Lebesgue outer measure)

Define $\mu^*: 2^{\mathbb{R}} \to [0, +\infty]$ by

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} b_i - a_i : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

"size of covering $\{(a_i, b_i]\}_{i=1}^{\infty}$"

$a_i \leq b_i$

Def: Given an outer measure μ^* on X, we say $A \subseteq X$ is μ^*-measureable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \quad \forall E \subseteq X$$

"A breaks apart all other sets nicely"

$$\mathcal{M}_{\mu^*} = \{ A \subseteq X : A \ \mu^*\text{-measureable} \}$$

Thm (Carathéodory): Given an outer measure μ^*,
(i) $\mathcal{M}_{\mu^*}$ is a σ-algebra
(ii) μ^* is a measure on $\mathcal{M}_{\mu^*}$.

We begin with some preliminary results...

Prop: For any outer measure μ^*, if $\mu^*(B)=0$, then $B \in \mathcal{M}_{\mu^*}$.

Prop:

(i) $\mathcal{M}_{\mu^*}$ is algebra

(ii) $\mu^*|_{\mathcal{M}_{\mu^*}}$ is finitely additive

↑ $E = X$

(iii) Given $\{B_i\}_{i=1}^n \subseteq \mathcal{M}_{\mu^*}$ disjoint,

$$\mu^*\left(E \cap \left(\bigcup_{i=1}^n B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

$\forall\ E \subseteq X$.

Prop:

Fix $E \subseteq X$, $\{B_i\}_{i=1}^\infty \in \mathcal{M}_{\mu^*}$ disjoint.

$$\mu^*(E) = \sum_{i=1}^\infty \mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^\infty B_i\right)^c\right).$$

Pf: "$\leq$" follows by subadditivity of μ^*. It remains to show the opposite inequality.

part (i)

Since $\mathcal{M}_{\mu^*}$ is an algebra, $\bigcup_{i=1}^{n} B_i \in \mathcal{M}_{\mu^*}$. So, $\forall E \subseteq X$,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{n} B_i\right)^c\right)$$

part (iii) ↓ (on $\mu^*(E \cap (\bigcup_{i=1}^{n} B_i))$)

monotonicity ↓ (on $\mu^*(E \cap (\bigcup_{i=1}^{n} B_i)^c)$)

$$\geq \sum_{i=1}^{n} \mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Sending $n \to +\infty$ gives the result!

Pf: (Carathéodory)

To show $\mathcal{M}_{\mu^*}$ is a σ-algebra, it remains to show $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{M}_{\mu^*}$ disjoint $\Rightarrow \bigcup_{i=1}^{\infty} B_i \in \mathcal{M}_{\mu^*}$.

For such B_i and any $E \subseteq X$,

$$\mu^*(E) = \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

subadd.

$$\geq \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) + \mu^*\left(E \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Thus $\bigcup_{i=1}^{\infty} B_i \in \mathcal{M}_{\mu^*}$.

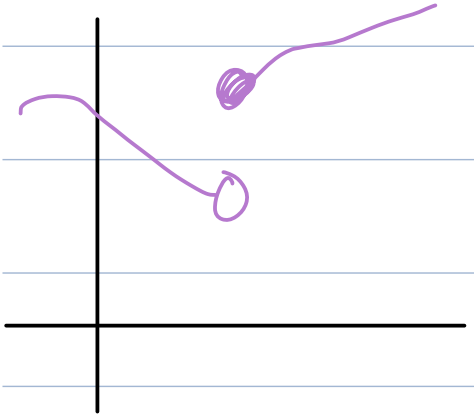
Finally, countable additivity follows from previous prop with $E = \bigcup_{i=1}^{\infty} B_i$. □

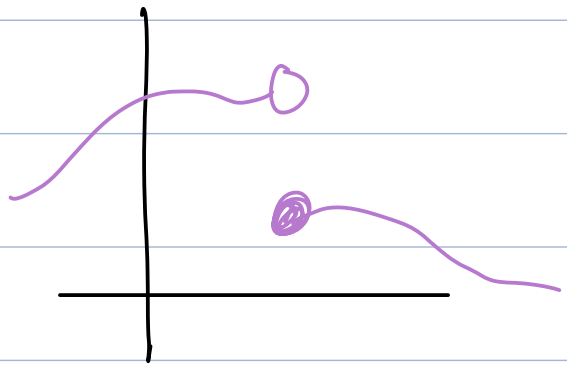
We return to a generalization of Lebesgue outer measure...

Recall: $F: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is right continuous if, $\forall x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(x) = F(x)$$

$$\| \quad \lim_{y \downarrow x} F(x)$$





Def: Given $F:\mathbb{R}\to\mathbb{R}$ nondecreasing, right cts, define $\mu_F^*:2^{\mathbb{R}}\to[0,+\infty]$ by

$$\mu_F^*(A) := \inf\left\{\sum_{i=1}^{\infty} \underbrace{F(b_i)-F(a_i)}_{|I_i|_F} : A\subseteq \bigcup_{i=1}^{\infty} \underbrace{(a_i,b_i]}_{I_i}\right\}$$

$a_i \le b_i$

These are Lebesgue-Stieltjes outer measures.

Ex: $F(x)=x$, we recover Lebesgue outer measure

Why these hypotheses on F?

Spoiler: We will show that for any finite measure μ on $\mathcal{B}_{\mathbb{R}}$ satisfies $\mu = \mu_F^*|_{\mathcal{B}_{\mathbb{R}}}$

for $F(x) = \mu((-\infty, x])$

↑ the cumulative distribution function of μ.

For F defined in this way,

$x \leq y \Rightarrow (-\infty, x] \subseteq (-\infty, y]$ &

$\Rightarrow F(x) \leq F(y)$

Likewise, for any sequence $x_n \searrow x$,

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n])$$

ctty from above ↓

$$= \mu\left(\bigcap_{n=1}^{\infty} (-\infty, x_n]\right)$$

$$= \mu((-\infty, x])$$

$$= F(x)$$

Rmk: (Optimization)

Given

- nonempty set X
- $f: X \to \mathbb{R} \cup \{+\infty\}$
- $C \subseteq X$, $C \neq \emptyset$

consider the optimization problem:

"objective function"

$$m := \inf\{f(x) : x \in C\}$$

"optimum"

"constraint set"

Recall: there exists a minimizing sequence, that is, a sequence x_n s.t. $\{x_n\}_{n=1}^{\infty} \subseteq C$ and

$$\lim_{n \to \infty} f(x_n) = m.$$

If $\exists\ x_* \in C$ s.t. $f(x_*) = m$, call x_* the optimizer.

Thm: For any $F: \mathbb{R} \to \mathbb{R}$ nondecreasing, right cts, μ_F^* is an outer measure on $\mathbb{R}$.

Thm: For $F: \mathbb{R} \to \mathbb{R}$ nondecreasing right cts, and all $a, b \in \mathbb{R}$, $a \leq b$,

$$\mu_F^*((a,b]) = F(b) - F(a).$$

It remains to show

$$A \subseteq \bigcup_{i=1}^{\infty} B_i \Rightarrow \mu_F^*(A) \leq \sum_{i=1}^{\infty} \mu_F^*(B_i).$$

Fix $A \subseteq \bigcup_{i=1}^{\infty} B_i$. WLOG $\sum_{i=1}^{\infty} \mu_F^*(B_i) < +\infty$.

Thus $\mu_F^*(B_i) < +\infty \quad \forall i \in \mathbb{N}$.

Fix $\varepsilon > 0$.

There exists $\{I_j^{i,\varepsilon}\}_{j=1}^{\infty}$ s.t.

- $B_i \subseteq \bigcup_{j=1}^{\infty} I_j^{i,\varepsilon}$
- $\mu_F^*(B_i) \leq \sum_{j=1}^{\infty} |I_j^{i,\varepsilon}|_F \leq \mu_F^*(B_i) + \frac{\varepsilon}{2^i}$

can't be a lower bound

Furthermore, $A \subseteq \bigcup_{i,j=1}^{\infty} I_j^{i,\varepsilon}$, so

$$\mu_F^*(A) \leq \sum_{i,j=1}^{\infty} |I_j^{i,\varepsilon}|_F \leq \sum_{i=1}^{\infty} \left(\mu_F^*(B_i) + \frac{\varepsilon}{2^i}\right)$$

$$= \left(\sum_{i=1}^{\infty} \mu_F^*(B_i)\right) + \varepsilon.$$

Pf: WLOG $a<b$.

"$\leq$" follows from defn of μ_F^* since $(a,b] \cup \emptyset \cup \emptyset \cup \ldots$ is an elt of constraint set

It remains to show "$\geq$".

It suffices to show $F(b)-F(a)$ is a lower bound.