Lecture 5 - Math 201a

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Recall:

Recall: $F: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is right continuous if, $\forall x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(x) = F(x)$$

Def: Given $F: \mathbb{R} \to \mathbb{R}$ nondecreasing, right cts, define $\mu_F^* : 2^{\mathbb{R}} \to [0, +\infty]$ by

$$\mu_F^*(A) := \inf\left\{ \sum_{i=1}^{\infty} \underbrace{F(b_i) - F(a_i)}_{|I_i|_F} : A \subseteq \bigcup_{i=1}^{\infty} \underbrace{(a_i, b_i]}_{I_i} \right\}$$

$a_i \leq b_i$

Rmk: (Optimization)

Given

- nonempty set X
- $f: X \to \mathbb{R} \cup \{+\infty\}$
- $C \subseteq X$, $C \neq \emptyset$

consider the optimization problem:

"objective function"

$$m := \inf\{f(x) : x \in C\}$$

"optimum"

"constraint set"

Recall: there exists a minimizing sequence, that is, a sequence x_n s.t. $\{x_n\}_{n=1}^{\infty} \subseteq C$ and

$$\lim_{n \to \infty} f(x_n) = m.$$

If $\exists\ x_* \in C$ s.t. $f(x_*) = m$, call x_* the optimizer.

Thm: For any $F: \mathbb{R} \to \mathbb{R}$ nondecreasing, right cts, μ_F^* is an outer measure on $\mathbb{R}$.

Thm: For $F: \mathbb{R} \to \mathbb{R}$ nondecreasing right cts, and all $a, b \in \mathbb{R}$, $a \leq b$,

$$\mu_F^*((a,b]) = F(b) - F(a).$$

Pf: WLOG $a<b$.

"$\leq$" follows from defn of μ_F^* since $(a,b] \cup \emptyset \cup \emptyset \cup \dots$ is an elt of constraint set

It remains to show "$\geq$".

It suffices to show $F(b)-F(a)$ is a lower bound for the set

$$\left\{ \sum_{i=1}^{\infty} F(b_i)-F(a_i) : (a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$

Fix a cover $\{(a_i, b_i]\}_{i=1}^{\infty}$ s.t.

$$(a,b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i].$$

Fix $\varepsilon > 0$ (arbitrary). It suffices to show

$$F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

Suppose ε is suff small s.t. $a + \varepsilon < b$.

Since F is right cts, $\forall i \in \mathbb{N}$, $\exists \delta_i > 0$ s.t.

$$F(b_i + \delta_i) \leq F(b_i) + \frac{\varepsilon}{2^i}$$

$$[a+\varepsilon, b] \subseteq (a, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \delta_i)$$

Since $[a+\varepsilon, b]$ is cpt, there exists a finite subcover; after reindexing, we have

$$[a+\varepsilon, b] \subseteq \bigcup_{i=1}^{N} (a_i, b_i + \delta_i)$$

We may discard any open interval not needed in the subcover.

This guarantees that, up to reindexing again,

$$b_1+\delta_1 < b_2+\delta_2 < \dots < b_N+\delta_N$$

Furthermore, for each $i=1,\dots,N-1$,

$$b_i+\delta_i \in (a_{i+1}, b_{i+1}+\delta_{i+1})$$

$a_1 < a_2 < a_3 \dots$

$$\begin{aligned} F(b)-F(a+\varepsilon) &\le F(b_N+\delta_N)-F(a_1) \\ &= F(b_N+\delta_N)-F(a_N) \\ &\le \sum_{i=1}^{N-1} F(a_{i+1})-F(a_i) \end{aligned}$$

$$F(b) - F(a+\varepsilon)$$

$$\vdots$$

$$\leq F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} F(b_i+\delta_i) - F(a_i)$$

$$= \sum_{i=1}^{N} F(b_i+\delta_i) - F(a_i)$$

$$\leq \sum_{i=1}^{N} F(b_i) - F(a_i) + \frac{\varepsilon}{2^i}$$

$$\leq \left(\sum_{i=1}^{\infty} F(b_i) - F(a_i) \right) + \varepsilon$$

Sending $\varepsilon \to 0$ gives the result. $\square$

Def: When $F(x)=x$, we write

$\lambda^* := \mu_F^*$ = Lebesgue outer measure

$\mathfrak{M}_{\lambda^*} = \mathfrak{M}_{\mu_F^*}$ = Lebesgue measurable sets

$\lambda = \lambda^*|_{\mathfrak{M}_{\lambda^*}}$ = Lebesgue measure

Def: For $F:\mathbb{R}\to\mathbb{R}$ nondecreasing, right cts,

$\mu_F := \mu_F^*|_{\mathfrak{M}_{\mu_F^*}}$ = Lebesgue-Stieltjes measure

Thm: For $F:\mathbb{R}\to\mathbb{R}$ nondecreasing, right cts,

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathfrak{M}_{\mu_F^*}.$$

Pf: It suffices to show, $\forall b \in \mathbb{R}$,

$$(-\infty, b] \in \mathcal{M}_{\mu_F^*},$$

that is, for all $E \subseteq \mathbb{R}$,

$$\mu_F^*(E) \geq \mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c)$$

WLOG, $\mu_F^*(E) < +\infty$.

Fix arbitrary $\varepsilon > 0$.

$\exists \{(a_i, b_i]\}_{i=1}^{\infty}$ s.t. $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$ and

$$\sum_{i=1}^{\infty} F(b_i) - F(a_i) \leq \mu_F^*(E) + \varepsilon.$$

Note that $b_i \wedge b = \min\{b_i, b\}$

$$(a_i, b_i] \cap (-\infty, b] \subseteq (a_i, b_i \wedge b]$$
$$(a_i, b_i] \cap (b, +\infty) \subseteq (a_i \vee b, b_i]$$

$$E \cap (-\infty, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i \wedge b]$$

$$E \cap (b, +\infty) \subseteq \bigcup_{i=1}^{\infty} (a_i \vee b, b_i]$$

Thus, $(s)_+ = \max\{s, 0\}$

$$\mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c)$$
$$\leq \sum_{i=1}^{\infty} \underbrace{(F(b_i \wedge b) - F(a_i))_+}_{s_i} + \underbrace{(F(b_i) - F(a_i \vee b))_+}_{t_i}$$

Case (i) if $b \leq a_i$, $s_i = 0$, $t_i = F(b_i) - F(a_i)$

(ii) if $b > b_i$, $s_i = F(b_i) - F(a_i)$, $t_i = 0$

(iii) if $b \in (a_i, b_i]$,

$s_i = F(b) - F(a_i)$, $t_i = F(b_i) - F(b)$

Thus, in all cases,

$s_i + t_i = F(b_i) - F(a_i)$.

Therefore,

$$\mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c)$$

$$\leq \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

$$\leq \mu_F^*(E) + \varepsilon$$

Sending $\varepsilon \to 0$ gives the result. □

The last item we hoped to have for Lebesgue measure is...

Thm: λ^* is translation invariant on $2^{\mathbb{R}}$. λ is translation invariant.

Pf: For any $a \in \mathbb{R}$, $A \subseteq \mathbb{R}$,

$$A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$$

$$\Leftrightarrow A + a \subseteq \bigcup_{i=1}^{\infty} (a_i + a, b_i + a]$$

Furthermore, both covers have the same "length" with respect to $\int F(x) = x$.

Thus $\lambda^*(A) = \lambda^*(A + a)$.

Since $\lambda = \lambda^* |_{\mathfrak{M}_{\lambda^*}}$, it remains to show, $\forall a \in \mathbb{R}$,

$$A \in \mathfrak{M}_{\lambda^*} \Rightarrow A + a \in \mathfrak{M}_{\lambda^*}$$

Suppose $A \in \mathfrak{M}_{\lambda^*}$.

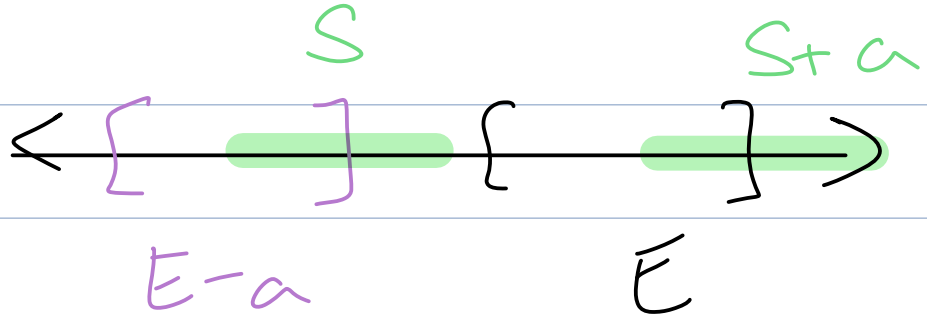
Fix $E \subseteq \mathbb{R}$. WTS

$$\lambda^*(E) \geq \lambda^*(E \cap (A+a)) + \lambda^*(E \cap (A+a)^c)$$

Note that

$$(E-a) \cap S = (E \cap (S+a)) - a$$

$$(S+a)^c = S^c + a$$



Since $A \in \mathfrak{M}_{\lambda^*}$,

$$\lambda^*(E) = \lambda^*(E-a)$$
$$\geq \lambda^*((E-a)\cap A) + \lambda^*((E-a)\cap A^c)$$
$$= \lambda^*([E\cap(A+a)]-a) + \lambda^*([E\cap(A^c+a)]-a)$$
$$= \lambda^*(E\cap(A+a)) + \lambda^*(E\cap(A+a)^c) \quad \square$$

