

Lecture 6 - Math 201a

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Midterm 1: Tuesday, October 20th

Practice Midterm posted today

HW due Thursday, Oct 22nd

Recall:

Def: When $F(x) = x$, we write
 $\lambda^* \subseteq \mu_F^* = \text{Lebesgue outer measure}$
 $m_{\lambda^*} = m_{\mu_F^*} = \text{Lebesgue measurable sets}$
 $\lambda = \lambda^*|_{m_{\lambda^*}} = \text{Lebesgue measure}$

Def: For $F: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing,
right cts,
 $\mu_F := \mu_F^*|_{m_{\mu_F^*}} = \text{Lebesgue-Stieltjes measure}$

Thm: For $F: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing,
right cts,
 $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F^*}$.

Thm: λ^* is translation invariant
on $2^{\mathbb{R}}$. λ is translation
invariant.

Now, we can show...

Thm: Suppose μ is a finite
Borel measure on $\mathbb{R}$ and
define

$$F(x) := \mu((-\infty, x]).$$

Then

- (i) F nondecreasing & right cts
- (ii) $\mu(E) = \mu_F(E) \quad \forall E \in \mathcal{B}_{\mathbb{R}}$

Pl:

We already showed (i).
It remains to show (ii).

Fix $a < b$. Then

$$\begin{aligned}\mu((a, b]) &= \mu((-\infty, b] \setminus (-\infty, a]) \\ &= \mu((-\infty, b]) - \mu((-\infty, a]) \\ &= F(b) - F(a) \\ &= \mu_F((a, b])\end{aligned}$$

by prev.
lecture



arbitrary

Fix $E \in \mathcal{B}_{\mathbb{R}}$ and a cover $\{(a_i, b_i]\}_{i=1}^{\infty}$ so that $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$.

Then, by subadditivity,

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu((a_i, b_i]) = \sum_{i=1}^{\infty} F(b_i) - F(a_i)$$

By defn of infimum as greatest lower bound,

$$\mu(E) \leq \mu_F^\circ(E) = \mu_F(E)$$

$$\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}_{\mu_F^\circ}$$

Since $E \in \mathcal{B}_\mathbb{R}$ was arbitrary,
 $\mu(E^c) \leq \mu_F(E^c)$

$$A_n := (-n, n]$$

Furthermore,

$$\mu(\mathbb{R}) = \mu\left(\bigcup_{n=1}^{\infty} (-n, n]\right)$$

↓ cty from below

$$= \lim_{n \rightarrow \infty} \mu((-n, n])$$

$$= \lim_{n \rightarrow \infty} \mu_F((-n, n])$$

↓ cty from below

$$= \mu_F(\mathbb{R}).$$

Thus,

$$\mu(E) = \mu(\mathbb{R}) - \mu(E^c) \geq \mu_F(\mathbb{R}) - \mu_F(E^c) = \mu_F(E).$$

Last major result for L-S
measures... regularity
(approximation)

Lemma: Given $F: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing
right cts, $\forall E \in \mathcal{M}_{\mu_F^+}$,

$$\mu_F(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu_F((a_i, b_i]) : E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}$$



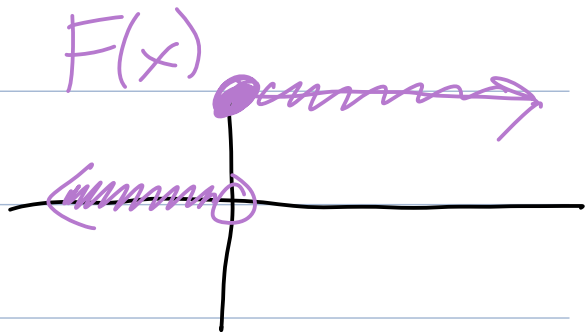
$\uparrow$
 $a_i < b_i$

Ex:

$$\mu = \delta_0$$

$$F(x) = \mu((-\infty, x])$$

$$\underbrace{\mu((-1, 0))}_0 \neq \underbrace{F(0) - F(-1)}_1$$



Pl:

On HW3 Q2, you will show " $\leq$ "

We will show " $\geq$ " WLOG,
assume $\mu_F(E) < +\infty$.

Fix $\varepsilon > 0$. By defn of μ_F^* , $\exists$
 $\{[a_i, b_i]\}_{i=1}^{\infty}$ s.t. $E \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i]$ and

$$\begin{aligned}\mu_F(E) + \varepsilon &\geq \sum_{i=1}^{\infty} F(b_i) - F(a_i) \\ &= \sum_{i=1}^{\infty} \mu_F([a_i, b_i])\end{aligned}$$

By cty from above, $\forall i$,

$$\lim_{n \rightarrow \infty} \mu_F([a_i, b_i + \frac{1}{n}]) = \mu_F([a_i, b_i])$$

For all i , choose δ_i s.t.

$$\mu_F(a_i, b_i + \delta_i) \leq \mu_F(a_i, b_i) + \frac{\varepsilon}{2^i}$$

Thus, $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \delta_i)$ and

$$\mu_F(E) + \varepsilon \geq \sum_{i=1}^{\infty} \mu_F(a_i, b_i + \delta_i)$$

$$\geq \sum_{i=1}^{\infty} \mu_F(a_i, b_i + \delta_i) - \frac{\varepsilon}{2^i}$$

$$= \sum_{i=1}^{\infty} \mu_F(a_i, b_i + \delta_i) - \varepsilon$$

$$\Rightarrow \mu_F(E) + 2\varepsilon \geq \sum_{i=1}^{\infty} \mu_F(a_i, b_i + \delta_i)$$

$$\geq \text{ghost}$$

Sending $\varepsilon \rightarrow 0$ gives " $\geq$ ".

Thm (Regularity of L-S measures)
 Suppose $F: \mathbb{R} \rightarrow \mathbb{R}$ nondecreasing,
 right cts. For all $E \in \mathcal{M}$, μ_F
 (*) "outer regularity"

$$\mu_F(E) = \inf \{ \mu_F(U) : E \subseteq U, U \text{ open} \}$$

$$= \sup \{ \mu_F(K) : K \subseteq E, K \text{ cpt} \}$$

(**) "inner regularity"

Rmk: A measure μ on
 $(X, \mathcal{B}(X))$ is a Radon measure
 if

- ① it is finite on cpt sets
 - ② outer regular
 - ③ inner regular
- on σ -finite sets

Thus, the theorem shows L-S
 measures are Radon measures.

Pl: Fix $E \in \mathcal{M}_{\mu_F}$.
First, show $(*)$.

Note that " $\leq$ " follows by monotonicity. We now show " $\geq$." wLOG $\mu_F(E) < +\infty$.
Fix $\varepsilon > 0$ arbitrary.

By prev lem, $\exists \{(a_i, b_i)\}_{i=1}^{\infty}$ s.t. $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$
and

$$\begin{aligned} \mu_F(E) + \varepsilon &\geq \sum_{i=1}^{\infty} \mu_F((a_i, b_i)) \\ &\quad \downarrow \text{subadd} \\ &\geq \mu_F\left(\bigcup_{i=1}^{\infty} (a_i, b_i)\right) \end{aligned}$$

$$\geq \inf \{ \mu_F(U) : E \subseteq U, U \text{ open} \}$$

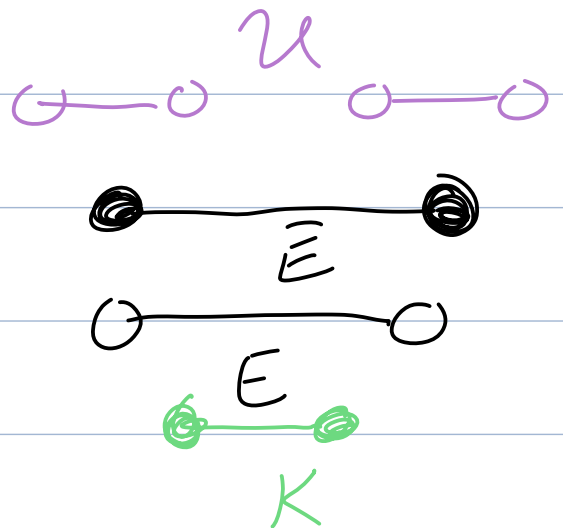
Sending $\varepsilon \rightarrow 0$ gives result.

We now show ~~($\star$)~~. " $\geq$ " follows by monotonicity. We show " $\leq$ ".

Case 1: Assume E is bounded. (So $\bar{E}$ is also bounded).

Fix $\varepsilon > 0$. By ~~($\star$)~~, $\exists \mathcal{U} \geq \bar{E} \setminus E$
 $\mathcal{U}$ open s.t.

 $\mu(\mathcal{U}) \leq \mu(\bar{E} \setminus E) + \varepsilon$



Define $K := \bar{E} \setminus \mathcal{U}$.
 K is compact.

Also,

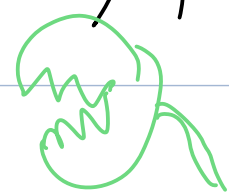
$$\begin{aligned} K &= \overline{E} \cap \mathcal{U}^c \subseteq \overline{E} \cap (\overline{E} \cap E^c)^c \\ &= \overline{E} \cap (\overline{E}^c \cup \overline{E}) \\ &= \overline{E} \end{aligned}$$

Furthermore,

$$\begin{aligned} \mu_F(E) &= \mu_F(E \cap \mathcal{U}) + \mu_F(E \setminus \mathcal{U}) \\ &\leq \mu_F(E \cap \mathcal{U}) + \mu_F(K) \end{aligned}$$

Thus, rearranging,

$$\sup_{\forall I} \{ \mu_F(K) : K \subseteq E, K \in \mathcal{I} \}$$

$$\begin{aligned} \mu_F(K) &\geq \mu_F(E) - \mu_F(E \cap \mathcal{U}) \\ &= \mu_F(E) - [\mu_F(\mathcal{U}) - \mu_F(\mathcal{U} \setminus E)] \\ &\geq \mu_F(E) - \mu_F(\mathcal{U}) + \mu_F(E \setminus \mathcal{U}) \\ &\geq \mu_F(E) - \varepsilon \end{aligned}$$


Sending $\varepsilon \rightarrow 0$ gives result.

Case 2: $E \subseteq \mathbb{R}$ unbd.

↓ bounded

$$E_j = E \cap (j, j+1], \quad j \in \mathbb{Z}$$

Fix $\varepsilon > 0$. By what we've already shown, $\exists$
 $K_j \subseteq E_j \cup K_j \text{ cpt, s.t.}$

$$\mu_F(K_j) \geq \mu_F(E_j) - \frac{\varepsilon}{2|j|}$$

Define $H_n = \bigcup_{j=-n}^n K_j$ ← disjoint union

Then $H_n \text{ cpt, } H_n \subseteq E$

For all $n \in \mathbb{N}$,

$$\mu_F(H_n) = \sum_{j=-n}^n \mu_F(K_j)$$

$$\geq \sum_{j=-n}^n \mu_F(E_j) - \frac{\varepsilon}{2} |j|$$

↓ sub
add

$$\geq \mu_F\left(\bigcup_{j=-n}^n E_j\right) - 3\varepsilon$$

(1) (2)

Suppose $\mu_F(E) < +\infty$.

By cty from below, we may choose n suff large so

$A_n = \bigcup_{j=-n}^n E_j$
 $\bigcup_{n=-\infty}^{\infty} A_n = \bigcup_{j=-\infty}^{\infty} E_j$

$$\sup_{|I|} \{ \mu_F(K) : K \subseteq E, K \cap I \neq \emptyset \}$$

$$\mu_F(H_n) \geq \mu_F(E) - 4\varepsilon$$

Sending $\varepsilon \rightarrow 0$ gives result.

Finally, if $\mu_F(E) = +\infty$, (1)
ensured

$$\sup\{\mu_F(K) : K \subseteq E, K \subset \mathbb{R}^n\} \\ \geq \lim_{n \rightarrow \infty} \mu_F(H_n) = +\infty,$$

So

$$\sup\{\mu_F(K) : K \subseteq E, K \subset \mathbb{R}^n\} \geq \mu_F(E). \quad \square$$

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Next class: from measures
to integration