Lecture 7 - Math 201a

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Midterm 1: Tuesday, October 20th

HW due Thursday, Oct 22nd

Thm: Suppose μ is a finite Borel measure on $\mathbb{R}$ and define

$$F(x) := \mu((-\infty, x]).$$

Then

(i) F nondecreasing + right cts

(ii) $\mu(E) = \mu_F(E) \; \forall \; E \in \mathcal{B}_{\mathbb{R}}$

Lemma: Given $F:\mathbb{R}\to\mathbb{R}$ nondecreasing right cts, $\forall E\in\mathcal{M}_{\mu_F}$,

$$\mu_F(E)=\inf\left\{\sum_{i=1}^{\infty}\mu_F((a_i,b_i)) : E\subseteq\bigcup_{i=1}^{\infty}(a_i,b_i)\right\}$$

($a_i<b_i$)

Thm (Regularity of L-S measures) Suppose $F:\mathbb{R}\to\mathbb{R}$ (nondec, right cts.

For $E\in\mathcal{M}_{\mu_F}$,

$$\mu_F(E)=\inf\{\mu_F(U): U \text{ open}, U\supseteq E\}$$

$$=\sup\{\mu_F(K): K \text{ cpt}, K\subseteq E\}$$

Integration

Given $f: X \to Y$, $E \subseteq Y$

$f^{-1}(E) := \{x \in X : f(x) \in E\}$

Recall from HW1:

- $f^{-1}\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f^{-1}(E_\alpha)$
- $f^{-1}\left(\bigcap_\alpha E_\alpha\right) = \bigcap_\alpha f^{-1}(E_\alpha)$
- $f^{-1}(E^c) = (f^{-1}(E))^c$

Suppose $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are measurable spaces. Then the following are σ-algebras:

$\{f^{-1}(E) : E \in \mathcal{N}\}$ "pullback of $\mathcal{N}$"

$\{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}$ "pushforward of $\mathcal{M}$"

Def: $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if, for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.

"the preimage of every measurable set is measureable"

$$\Updownarrow$$

$$\{f^{-1}(E) : E \in \mathcal{N}\} \subseteq \mathcal{M}$$

$$\Updownarrow$$

$$\{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{N}$$

Important case

$f: X \to \overline{\mathbb{R}}$

When $f: X \to \mathbb{R}$, we will suppose the range is endowed with $\mathcal{B}_{\mathbb{R}}$.

$\mathcal{B}_{\overline{\mathbb{R}}}$

Def: $f: \mathbb{R} \to \mathbb{R}$ is **Lebesgue measureable** if it is $(\mathcal{M}_{\lambda^*}, \mathcal{B}_{\mathbb{R}})$-measureable

Given X, Y topological spaces, $f: X \to Y$ is **Borel measureable** if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Rmk: $f: \mathbb{R} \to \mathbb{R}$.

f is Borel measurable $\Longrightarrow$ f is Lebesgue measureable

We will see on HW4 that $\mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{M}_{\lambda^*}$.

Prop: Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ where $\mathcal{N}$ is generated by $\mathcal{E} \subseteq 2^Y$, then

- f is $(\mathcal{M}, \mathcal{N})$-measurable
- $\Leftrightarrow \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{M}$.

Pf: "$\Downarrow$" follows immediately from defn.

We now show "$\Uparrow$."
By hypothesis,
$$\mathcal{E} \subseteq \{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}$$

Since $\mathcal{M}(\mathcal{E}) = \mathcal{N}$ and pushforward is a σ-algebra,

$$\mathcal{N} \subseteq \{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}. \quad \square$$

Pf: On HW3, you show $\{(a,+\infty]: a\in\mathbb{R}\}$ generate $\mathcal{B}_{\overline{\mathbb{R}}}$.

"$\Downarrow$" follows immediately from defn.

"$\Uparrow$" follows from propositim.

$=$

Example: If $(X,\mathcal{M})$ is a measureable space, define $f: X\to\overline{\mathbb{R}}$ by

$$f(x)=c \quad \forall\, x\in X$$

for $c\in\overline{\mathbb{R}}$, then f is $(\mathcal{M},\mathcal{B}_{\overline{\mathbb{R}}})$-measureable.

Cor: If X and Y are topological spaces, then every continuous function is Borel measureable.

Pf: Since f cts, $\forall\, U \subseteq Y$, U open, $f^{-1}(U)$ is open, hence $f^{-1}(U) \in \mathcal{B}_X$.

Since $\mathcal{E} := \{U \subseteq Y,\ U \text{ open}\}$ generates $\mathcal{B}_Y$, the result follows from proposition.

Cor: If $(X, \mathcal{M})$ is a measureable space and $f: X \to \overline{\mathbb{R}}$ then

f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measureable
$\Leftrightarrow$ $f^{-1}((a, +\infty]) \in \mathcal{M}$ $\forall\, a \in \mathbb{R}$.

$$f^{-1}((a,+\infty]) = \begin{cases} \chi & \text{if } c > a \\ \emptyset & \text{if } c \leq a \end{cases}$$

We always have $\chi, \emptyset \in \mathcal{M}$.

Thm: Fix a measurable space $(\chi, \mathcal{M})$ and $\{f_i\}_{i=1}^{\infty}$, s.t. $f_i : \chi \to \overline{\mathbb{R}}$ and f_i is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Then the following functions are also $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable:

convention

(i) $f_1 + f_2 \leftarrow (+\infty) + (-\infty) = 0$

(ii) $f_1 \cdot f_2 \leftarrow 0 \cdot (\pm\infty) = 0$

(iii) $f_1 \vee f_2$

(iv) $f_1 \wedge f_2$

(v) $\sup_i f_i$

(vi) $\inf_i f_i$

(vii) $\limsup_i f_i$

(viii) $\liminf_i f_i$

(ix) $\lim_{i\to\infty} f_i$, as long as $\lim_{i\to\infty} f_i(x)$ exists for all $x \in X$.

Note: $(f \vee g)(x) = \max\{f(x), g(x)\}$

$(f \wedge g)(x) = \min\{f(x), g(x)\}$

Pf:

(i) + (ii): HW4

(iii) $(f_1 \vee f_2)^{-1}((a, +\infty])$

$= \{x \in X : (f_1 \vee f_2)(x) > a\}$

$= \{x \in X : f_1(x) > a\} \cup \{x \in X : f_2(x) > a\}$

$= f_1^{-1}((a, +\infty]) \cup f_2^{-1}((a, +\infty])$

Since f_1, f_2 measurable, this belongs to $\mathcal{M}$ $\forall a \in \mathbb{R}$.

(iv) $f_1 + f_2 - f_1 \vee f_2 = f_1 \wedge f_2$

sum of meas is meas by (i)

$(-1) \cdot (f_1 \vee f_2)$ is meas by (ii)

(v) $\left(\sup_i f_i\right)^{-1}((a, +\infty])$

$= \{x \in X : \sup_i f_i(x) > a\}$

$= \{x \in X : \exists\, i \text{ s.t. } f_i(x) > a\}$

$= \bigcup_{i=1}^{\infty} \{x \in X : f_i(x) > a\}$

$= \bigcup_{i=1}^{\infty} f_i^{-1}((a, +\infty]) \in \mathcal{M}$ since $\{f_i\}_{i=1}^{\infty}$ meas.

Aside:

$g : \mathbb{R}^2 \to \mathbb{R}$ cts

$f_1, f_2 : X \to \mathbb{R}$ meas

$g \circ (f_1, f_2)$ is meas?

(vi) follows from previous parts by multiplication by (-1) ☺

(vii) $\limsup_{i\to\infty} f_i = \inf_{k\in\mathbb{N}} \sup_{i\geq k} f_i$

follows from (v), (vi)

(viii) same as previous part...

(ix) if $\lim_{i\to\infty} f_i$ exists $\forall x \in X$,

$\lim_{i\to\infty} f_i = \limsup_{i\to\infty} f_i$, so

follows from (vii).

Remark: Consider $f, g : \mathbb{R} \to \mathbb{R}$

f Borel meas
g Borel meas
$\Rightarrow$ $f \circ g$ Borel meas

CORRECTION

$$(f \circ g)^{-1}(E) = g^{-1}(f^{-1}(E))$$

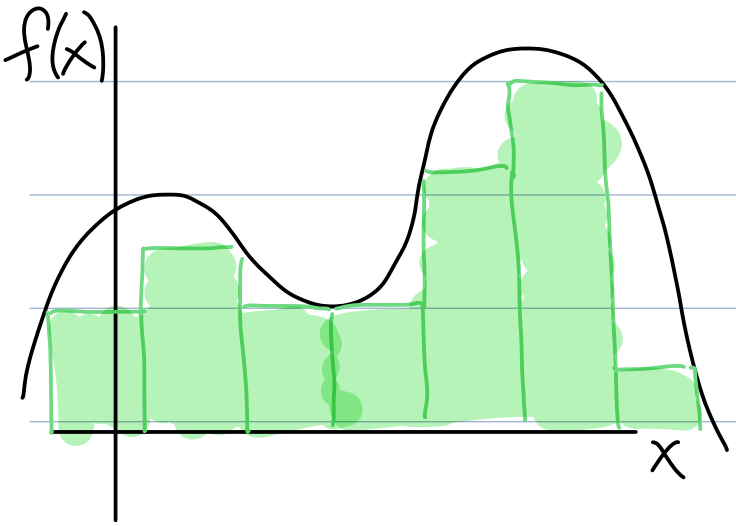
$$\{x : f \circ g(x) \in E\} = \{x : g(x) \in f^{-1}(E)\}$$

~~f Lebesgue meas
g Borel meas
$\Rightarrow$ $f \circ g$ Lebesgue meas~~

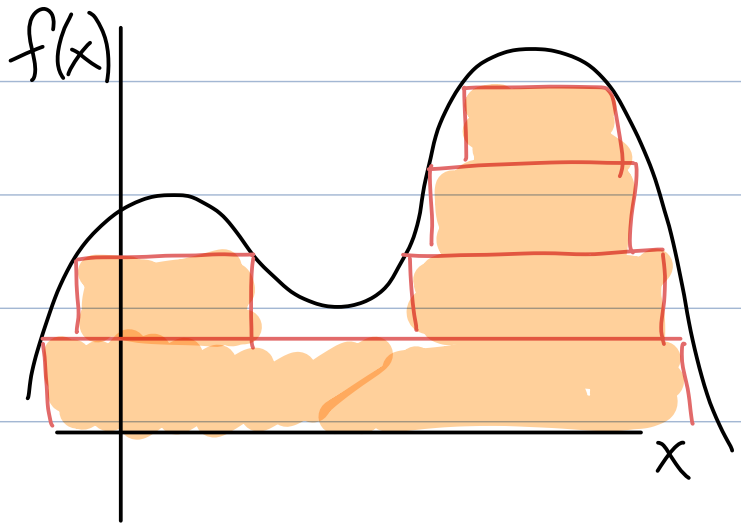
f Borel meas
g Lebesgue meas
$\Rightarrow$ $f \circ g$ Lebesgue meas

$G \in \mathcal{B}_{\mathbb{R}} \Rightarrow g^{-1}(G) \in \mathcal{M}_{\lambda} \to E \in \mathcal{B}_{\mathbb{R}}$.
$F \in \mathcal{B}_{\mathbb{R}} \Rightarrow f^{-1}(F) \in \mathcal{B}_{\mathbb{R}}$ $(f \circ g)^{-1}(E) \in \mathcal{M}_{\lambda}$

Key idea of Lebesgue integral:



Riemann

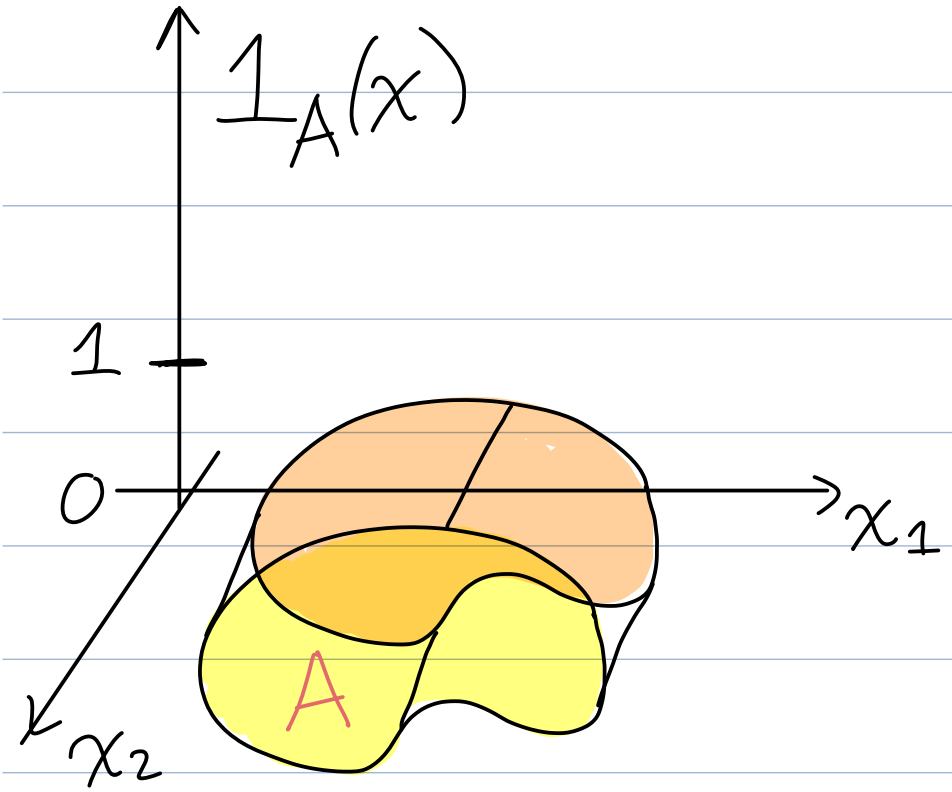


Lebesgue

Simple functions

Def: For any $A \subseteq X$, the indicator function of A is

$$1_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$



When is an indicator fn measureable?

$$1_A^{-1}(B) = \begin{cases} \emptyset & 0,1 \notin B \\ A & 1 \in B, 0 \notin B \\ A^c & 0 \in B, 1 \notin B \\ X & 0,1 \in B \end{cases}$$

Thus if $A \in \mathcal{M}$, 1_A is $(\mathcal{M}, \mathcal{B}_R)$-measureable.