Lecture 8 - Math 201a

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HW4 Due Tuesday, Oct 27

Integration

Suppose $(X, \mathcal{M})$, $(Y, \mathcal{N})$ are measurable spaces. Then the following are σ-algebras:

$\{f^{-1}(E) : E \in \mathcal{N}\}$ "pullback of $\mathcal{N}$"

$\{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\}$ "pushforward of $\mathcal{M}$"

Def: $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if, for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.

"the preimage of every measurable set is measureable"

$$\Updownarrow$$

$$\{f^{-1}(E) : E \in \mathcal{N}\} \subseteq \mathcal{M}$$

$$\Updownarrow$$

$$\{E \subseteq Y : f^{-1}(E) \in \mathcal{M}\} \supseteq \mathcal{N}$$

Important case

$f: X \to \overline{\mathbb{R}}$

When $f: X \to \mathbb{R}$, we will suppose the range is endowed with $\mathcal{B}_{\mathbb{R}}$.

$\mathcal{B}_{\overline{\mathbb{R}}}$

Def: $f: \mathbb{R} \to \mathbb{R}$ is *Lebesgue measureable* if it is $(\mathcal{M}_{\lambda^*}, \mathcal{B}_{\mathbb{R}})$-measureable

Given X, Y topological spaces, $f: X \to Y$ is *Borel measureable* if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Prop: Given measurable spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ where $\mathcal{N}$ is generated by $\mathcal{E} \subseteq 2^Y$, then

- f is $(\mathcal{M}, \mathcal{N})$-measureable
- $\Updownarrow$
- $\forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{M}$.

Cor: If X and Y are topological spaces, then every continuous function is Borel measureable.

Cor: If $(X, \mathcal{M})$ is a measureable space and $f: X \to \overline{\mathbb{R}}$ then

f is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measureable

$\Updownarrow$

$f^{-1}((a, +\infty]) \in \mathcal{M} \quad \forall a \in \overline{\mathbb{R}}$.

Note: $(f \vee g)(x) = \max\{f(x), g(x)\}$

$(f \wedge g)(x) = \min\{f(x), g(x)\}$

Thm: Fix a measureable space $(X, \mathcal{M})$ and $\{f_i\}_{i=1}^{\infty}$, s.t. $f_i : X \to \overline{\mathbb{R}}$ and f_i is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Then the following functions are also $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable:

convention

(i) $f_1 + f_2$ $\leftarrow (+\infty) + (-\infty) = 0$

(ii) $f_1 \cdot f_2$ $\leftarrow 0 \cdot (\pm\infty) = 0$

(iii) $f_1 \vee f_2$

(iv) $f_1 \wedge f_2$

(v) $\sup_i f_i$

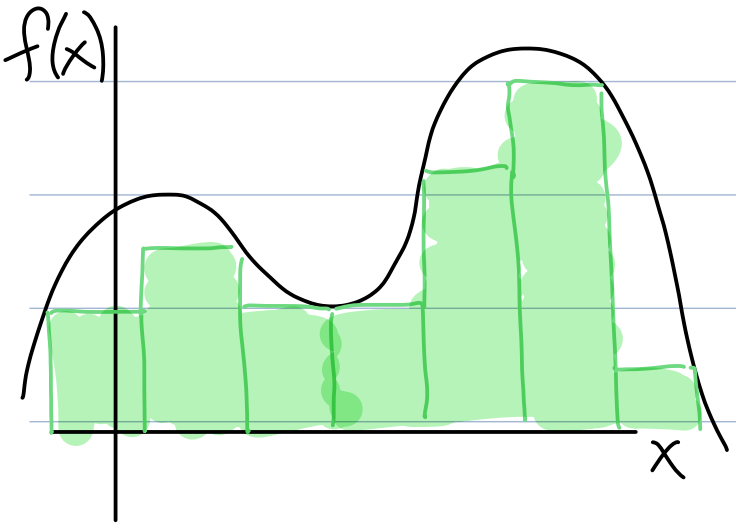
(vi) $\inf_i f_i$

(vii) $\limsup_i f_i$

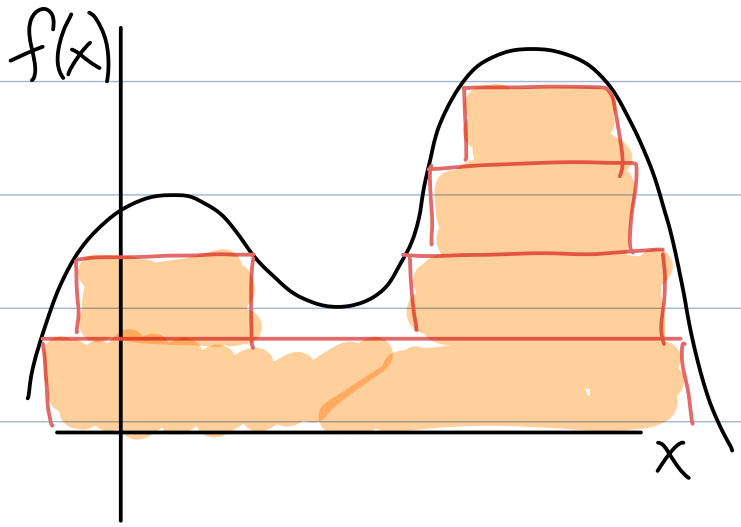
(viii) $\liminf_i f_i$

(ix) $\lim_{i \to \infty} f_i$, as long as $\lim_{i \to \infty} f_i(x)$ exists for all $x \in X$.

Key idea of Lebesgue integral:



Riemann

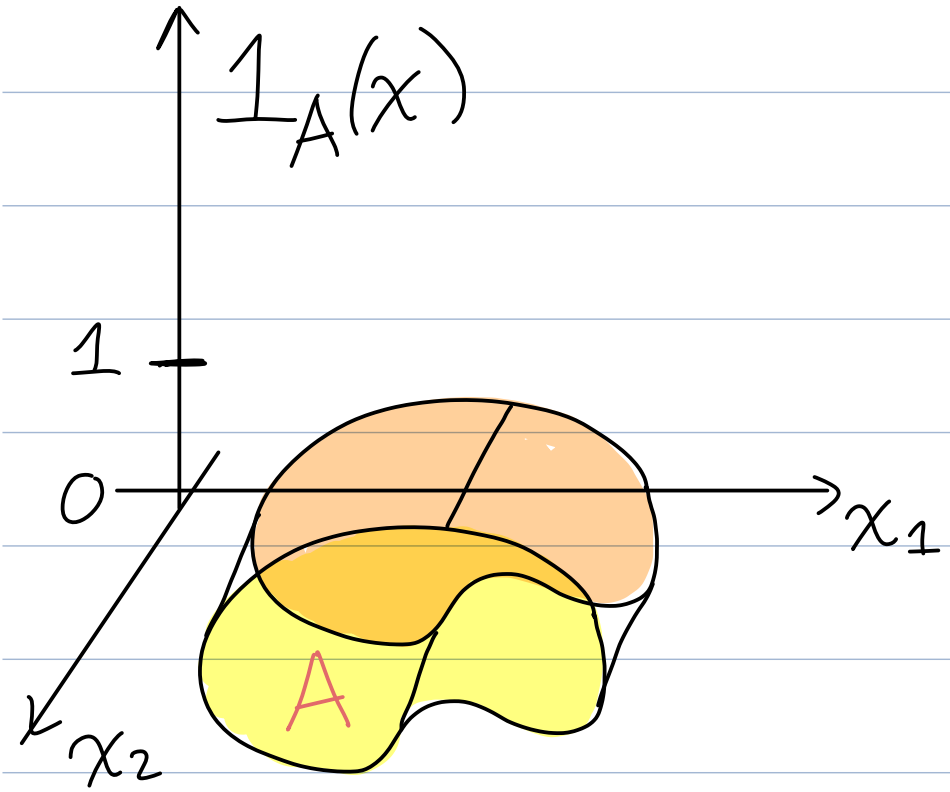


Lebesgue

Simple functions

Def: For any $A \subseteq X$, the indicator function of A is

$$1_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$$



Thus, $A \in \mathcal{M} \Leftrightarrow 1_A$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

Def: A $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable function $f: X \to \mathbb{R}$ is a simple function if $f(X)$ is a finite set.

The standard representation is

$$f(x) = \sum_{i=1}^{n} c_i 1_{E_i}(x)$$

where $f(x) = \{c_1, c_2, \ldots, c_n\}$ (distinct)

$E_i = f^{-1}(c_i)$

note that $\{E_i\}_{i=1}^{n}$ form a partition of X

Rmk (standard rep annoying)

Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2$

$f(x) = 2 1_{\mathbb{R}}(x) = 2 1_{(-\infty, 0]}(x) + 2 1_{(0, \infty)}(x)$

standard rep

Def: Given a measure space $(X, \mathcal{M}, \mu)$ and nonnegative simple function f with standard representation

$$f(x) = \sum_{i=1}^{n} c_i 1_{E_i}(x)$$

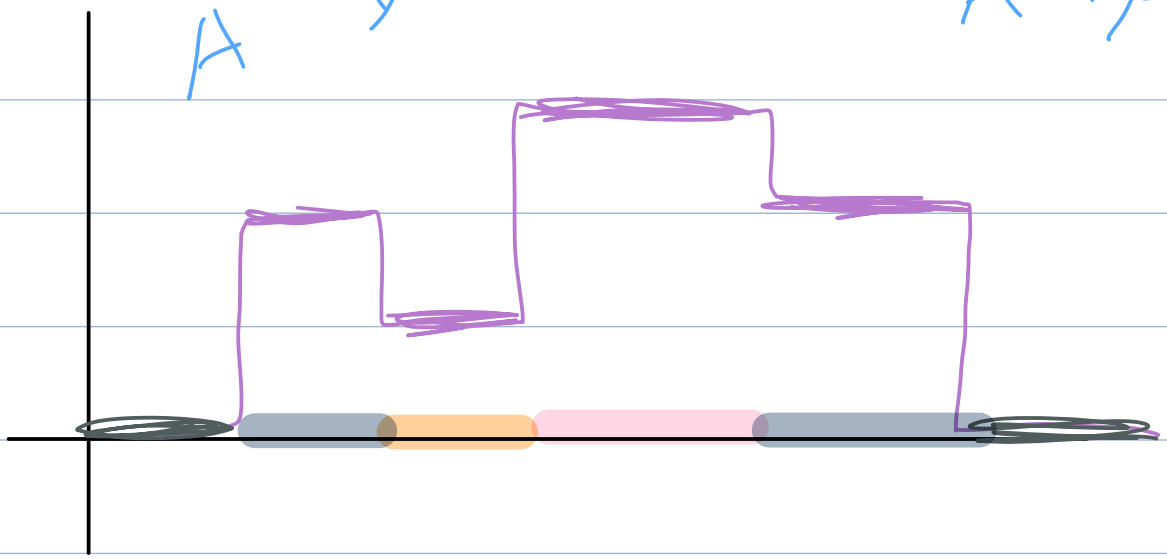
we define

$0 \cdot +\infty = 0$
(so $\int 0 \, d\mu = 0$)

$$\int f \, d\mu := \sum_{i=1}^{n} c_i \mu(E_i)$$

and, for $A \in \mathcal{M}$,

$$\int_A f \, d\mu := \int f \cdot 1_A \, d\mu$$



Rmk: $f(x) = \begin{cases} 1 & \text{if } x \in [0, +\infty) \\ -1 & \text{if } x \in (-\infty, 0) \end{cases}$

Wish could define

$$\int f \, d\lambda = (1) \cdot \lambda([0, +\infty)) + (-1) \cdot \lambda((-\infty, 0))$$

$$= \ddots$$

We will deal with this later...
for now, only integrate nonnegative functions.

Rmk: Notation for integrals

$$\int f d\mu = \int f(x) d\mu(x) = \int_X f(x) d\mu(x)$$

$$\int_A f d\mu = \int_A f(x) d\mu(x)$$

$\to f(x) \le g(x) \ \forall x \in X$

again, $0 \cdot (+\infty) = 0$

Prop: Given a measure space $(X, \mathcal{M}, \mu)$ and simple functions $f, g : X \to [0, +\infty)$

1. if $c \ge 0$, $\int cf d\mu = c \int f d\mu$
2. $\int (f+g) d\mu = \int f d\mu + \int g d\mu$
3. $f \le g \implies \int f d\mu \le \int g d\mu$

④ The function $A \mapsto \int_A f d\mu$

is a measure on $(X, \mathcal{M})$.

Rmk: Parts ① and ② ensure defn of $\int f d\mu$ is independent of how we represent f as a (finite) linear comb. of indicator fns. standard rep

That is, if $f = \sum_{i=1}^{n} c_i 1_{E_i} = \sum_{j=1}^{m} d_j 1_{F_j}$

then

$$\int f d\mu \overset{\text{def}}{=} \sum_{i=1}^{n} c_i \mu(E_i)$$

$$\| $$

$$\int \sum_{j=1}^{m} d_j 1_{F_j} d\mu \overset{②}{=} \sum_{j=1}^{m} \int d_j 1_{F_j} d\mu \overset{①}{=} \sum_{j=1}^{m} d_j \mu(F_j)$$

Pf:

① If $c=0$, then $cf=0$, so $\int cf\,d\mu = 0\cdot\mu(X)=0$.

OTOH, for any $\int f\,d\mu \in [0,+\infty]$, we take convention $c\int f\,d\mu = 0$.

If $c>0$, let $f=\sum_{i=1}^{n} c_i 1_{E_i}$ (standard).
Then $cf=\sum_{i=1}^{n} cc_i 1_{E_i}$ (standard).

Thus, defn of integral,

$$\int cf\,d\mu = \sum_{i=1}^{n} cc_i\,\mu(E_i)$$

$$= c\sum_{i=1}^{n} c_i\,\mu(E_i) = c\int f\,d\mu.$$

② Let standard

$$f=\sum_{i=1}^{n} c_i 1_{E_i},\quad g=\sum_{j=1}^{m} d_j 1_{F_j}$$

Since $\{E_i\}_{i=1}^{n}$, $\{F_j\}_{j=1}^{m}$ partition χ,

$$E_i=\bigcup_{j=1}^{m} E_i\cap F_j,\quad F_j=\bigcup_{i=1}^{n} F_j\cap E_i$$

Thus,

$$\int f d\mu+\int g d\mu$$

$$=\sum_i c_i \mu(E_i)+\sum_j d_j \mu(F_j)$$

$$=\sum_{i,j} c_i \mu(E_i\cap F_j)+\sum_{i,j} d_j \mu(E_i\cap F_j)$$

$$=\sum_{i,j}(c_i+d_j)\mu(E_i\cap F_j)$$

$$=\sum_{\ell=1}\ \sum_{i,j:\, c_i+d_j=\alpha_\ell}(c_i+d_j)\mu(E_i\cap F_j)$$

$$= \sum_{\ell=1}^{k} \alpha_\ell \, \mu(G_\ell).$$

$$= \int (f+g)\, d\mu$$

$(f+g)(\chi) = \{\alpha_1, \ldots, \alpha_k\}$ ← distinct

$\alpha_\ell = c_i + d_j$ for some i, j

$G_\ell := (f+g)^{-1}(\alpha_\ell)$.

③ Suppose $f \le g$ and

$$f = \sum_{i=1}^{n} c_i 1_{E_i}, \quad g = \sum_{j=1}^{m} d_j 1_{F_j}.$$

Then whenever $E_i \cap F_j \neq \emptyset$, $c_i \le d_j$. Thus,

$$\int f\,d\mu = \sum_i c_i\,\mu(E_i)$$

$$= \sum_{ij} c_i\,\mu(E_i \cap F_j)$$

$$\leq \sum_{ij} d_j\,\mu(E_i \cap F_j)$$

$$= \sum_j d_j\,\mu(F_j)$$

$$= \int g\,d\mu$$

(4) Let $\nu(A) = \int_A f\,d\mu$.

This is a nonnegative function on $\mathcal{M}$.

$$\nu(\emptyset) = \int_\emptyset f\,d\mu = \int f \cdot 1_\emptyset\,d\mu = 0$$ (since $\mu(\emptyset) = 0$)

Finally, for $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$ disjoint,

$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \int f 1_{\bigcup_{k=1}^{\infty} A_k} \, d\mu$$

$$= \int \sum_{i=1}^{n} c_i 1_{E_i} 1_{\bigcup_{k=1}^{\infty} A_k} \, d\mu$$

$$= \sum_{i=1}^{n} c_i \int 1_{E_i \cap \left[\bigcup_{k=1}^{\infty} A_k\right]} \, d\mu$$

$$= \sum_{i=1}^{n} c_i \, \mu\left(E_i \cap \left[\bigcup_{k=1}^{\infty} A_k\right]\right)$$

disjoint $\rightarrow$ $= \bigcup_{k=1}^{\infty} E_i \cap A_k$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{n} c_i \, \mu(E_i \cap A_k)$$

$$= \sum_{k=1}^{\infty} \int f 1_{A_k} \, d\mu$$

$$= \sum_{k=1}^{\infty} \nu(A_k)$$

$\square$

Integration of nonnegative measureable functions

Def: Given $(X, \mathcal{M}, \mu)$, $f: X \to [0, +\infty]$ measureable,

$$\int f \, d\mu := \sup\left\{ \int \varphi \, d\mu : \varphi \text{ simple}, 0 \le \varphi \le f \right\}$$

Rmk:

(i) If f is simple, this agrees w/ previous defn. $0 \cdot (+\infty) = 0$

(ii) If $c = 0$, $\int cf \, d\mu = c \int f \, d\mu$

If $c > 0$...

$$\int cf \, d\mu = \sup\left\{ \int \varphi \, d\mu : \varphi \text{ simple}, 0 \le \varphi \le cf \right\}$$

$$= \sup\left\{ \int \frac{\varphi}{c} \, d\mu : \frac{\varphi}{c} \text{ simple}, 0 \le \frac{\varphi}{c} \le f \right\}$$

$$= \sup\left\{\int \psi \, d\mu : \psi \text{ simple}, 0 \le \psi \le f\right\}$$

$$= c \sup\left\{\int \psi \, d\mu : \psi \text{ simple}, 0 \le \psi \le f\right\}$$

$$= c \int f \, d\mu$$

(iii) If $f \le g$, then $\int f \, d\mu \le \int g \, d\mu$ since constraint set for f contained in constraint set for g.

Major motivation for going beyond Riemann integral was difficulty obtaining weak enough conditions to ensure

$$\int \lim_{n\to\infty} f \, d\mu = \lim_{n\to\infty} \int f \, d\mu .$$

Our first example of such a result for Lebesgue integral...

Thm: (Monotone Convergence) Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measureable s.t. $f_n \leq f_{n+1}$ $\forall n$,

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu .$$