Lecture 9 - Math 201a

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HW5 due Sunday, November 1st

Def: A $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ measurable function $f: X \to \mathbb{R}$ is a simple function if $f(X)$ is a finite set.

The standard representation is

$$f(x) = \sum_{i=1}^{n} c_i 1_{E_i}(x)$$

where $f(X) = \{c_1, c_2, \dots, c_n\}$ (distinct)

$E_i = f^{-1}(c_i)$

note that $\{E_i\}_{i=1}^n$ form a partition of X

Def: Given a measure space $(X, \mathcal{M}, \mu)$ and nonnegative simple function f with standard representation

$$f(x) = \sum_{i=1}^{n} c_i 1_{E_i}(x)$$

we define

$0 \cdot +\infty = 0$ (so $\int 0 \, d\mu = 0$)

$$\int f \, d\mu := \sum_{i=1}^{n} c_i \mu(E_i)$$

and, for $A \in \mathcal{M}$,

$$\int_A f \, d\mu := \int f \cdot 1_A \, d\mu$$

Rmk: Notation for integrals

$$\int f d\mu = \int f(x) d\mu(x) = \int_X f(x) d\mu(x)$$

$$\int_A f d\mu = \int_A f(x) d\mu(x)$$

$\to f(x) \le g(x) \ \forall x \in X$

Prop: Given a measure space $(X, \mathcal{M}, \mu)$ and simple functions $f, g : X \to [0, +\infty)$

1. if $c \ge 0$, $\int cf d\mu = c \int f d\mu$
2. $\int (f+g) d\mu = \int f d\mu + \int g d\mu$
3. $f \le g \implies \int f d\mu \le \int g d\mu$
4. The function $A \mapsto \int_A f d\mu$

is a measure on $(X, \mathcal{M})$.

again, $0 \cdot (+\infty) = 0$

Integration of nonnegative measureable functions

Def: Given $(X, \mathcal{M}, \mu)$, $f: X \to [0, +\infty]$ measureable,

$$\int f \, d\mu := \sup\left\{ \int \varphi \, d\mu : \varphi \text{ simple}, 0 \le \varphi \le f \right\}$$

Rmk:

(i) If f is simple, this agrees w/ previous defn. $0 \cdot (+\infty) = 0$

(ii) If $c \ge 0$, $\int cf \, d\mu = c \int f \, d\mu$

(iii) If $f \le g$, then $\int f \, d\mu \le \int g \, d\mu$

Thm: (Monotone Convergence)
Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measureable s.t. $f_n \leq f_{n+1}$ $\forall n$,

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

Pf:

Since $f_n \leq f_{n+1}$ $\forall n$,

- $\lim_{n\to\infty} f_n(x)$ exists $\forall x \in X$
- $\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \Rightarrow \lim_{n\to\infty} \int f_n \, d\mu$ exists

To see "$\leq$", note that $f_n \leq \lim_{k\to\infty} f_k$, so $\int f_n \, d\mu \leq \int \lim_{k\to\infty} f_k \, d\mu$ $\forall n \in \mathbb{N}$.

We now show "$\geq$".

It suffices to show $\lim_{n\to\infty}\int f_n d\mu$ is an upper bound for

$$\left\{\int \varphi\, d\mu : \varphi \text{ simple}, 0 \le \varphi \le \lim_{n\to\infty} f_n\right\}$$

Choose φ simple s.t. $0 \le \varphi \le \lim_{n\to\infty} f_n$.

Fix $a \in (0,1)$ arbitrary.

(★) If $\varphi(x) \neq 0$, $a\varphi(x) < \lim_{n\to\infty} f_n(x)$

Define $E_n := \{x : f_n(x) \ge a\varphi(x)\}$ $\in \mathcal{M}$

Aside on measurability:

$E_n := \{x : f_n(x) - a\varphi(x) \ge 0\}$

$= (f_n - a\varphi)^{-1}([0, +\infty])$

- Since $f_n \le f_{n+1}$, $E_1 \subseteq E_2 \subseteq \dots$
- Furthermore $\bigcup_{n=1}^{\infty} E_n = X$, since
  - if $\varphi(x) \neq 0$, (★) $\exists N$ s.t. $n \ge N$ ensures $x \in E_n$,
  - if $\varphi(x) = 0$, $x \in E_n$, $\forall n \in \mathbb{N}$.

Therefore

$f_n \geq 1_{E_n} f_n$ $\quad$ $f_n 1_{E_n} \geq a\varphi$

$$\int f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} a\varphi \, d\mu = a \int_{E_n} \varphi \, d\mu$$

Sending $n \to +\infty$...

$A \mapsto \int_A \varphi \, d\mu$ is a measure

continuity from below

$$\lim_{n\to\infty} \int f_n d\mu \geq \lim_{n\to\infty} a \int_{E_n} \varphi \, d\mu = a \int_X \varphi \, d\mu$$

Sending $a \nearrow 1$ gives that

$$\lim_{n\to\infty} \int f_n d\mu \geq \int \varphi \, d\mu.$$

To prove things like $\int f+g\,d\mu = \int f\,d\mu + \int g\,d\mu$ we need to approximate nonneg. meas. fns by simple fns.

Thm: Given $f: X \to [0, +\infty]$ measureable, there exists a sequence fn of nonnegative simple fns s.t. $f_n \nearrow f$ pointwise. $\iff f_n \le f_{n+1}$ and $\lim_{n\to\infty} f_n = f$

Pf: For $n \in \{0, 1, 2, \ldots\}$, chop up the range of f (up to height 2^n) in increments of height 2^{-n}.

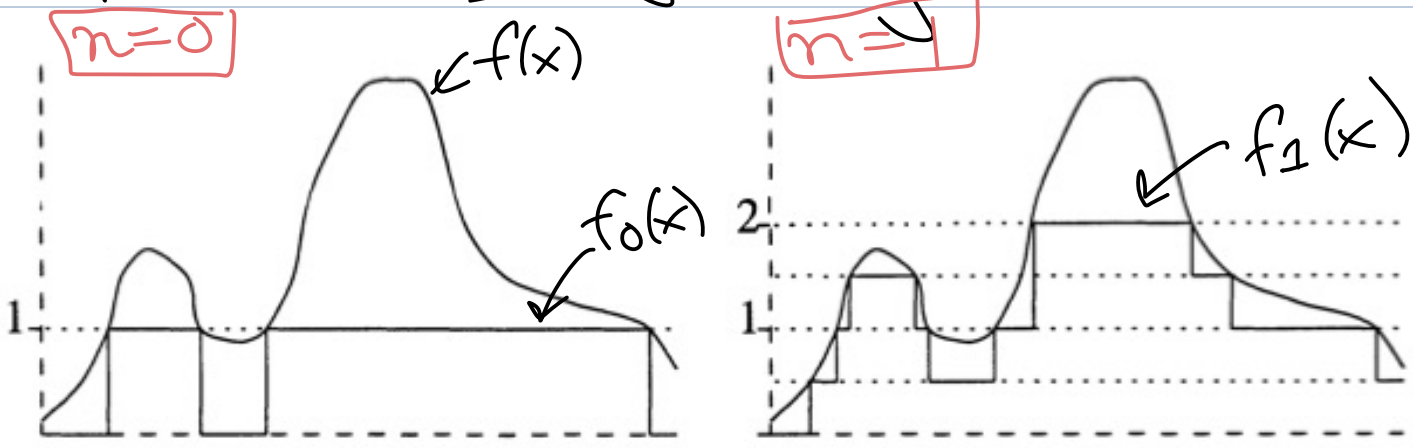


Fig. 2.1 The functions ϕ_0 (left) and ϕ_1 (right) in the proof of Theorem 2.10a.

$$F_n = f^{-1}((2^n, +\infty])$$
$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}])$$
for $k = 0, \dots, 2^{2n}-1$

$$f_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} 1_{E_n^k} + 2^n 1_{F_n}$$

Key properties

- $f_n(x) \leq f_{n+1}(x) \quad \forall x \in X$
- $0 \leq f(x) - f_n(x) \quad \forall x \in X$
- $f(x) - f_n(x) \leq 2^{-n} \quad \forall x \in F_n^c$

If $x \in \bigcup_{n=1}^{\infty} F_n^c$, then $\exists N$ s.t. $x \in F_n^c \ \forall n \geq N$, so

$$0 \leq f(x) - f_n(x) \leq 2^{-n} \ \forall n \geq N.$$

Hence $\lim_{n\to\infty} f_n(x) = f(x)$.

If $x \in \bigcap_{n=1}^{\infty} F_n$, then $f(x) = +\infty$

and $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 2^n = +\infty = f(x)$.

□

Thm (Beppo-Levi) Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measurable, then

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Pf: First fix $f, g : X \to [0, +\infty]$ meas. By prev Thm, $\exists \{\varphi_i\}_{i=0}^{\infty}, \{\psi_i\}_{i=1}^{\infty}$ simple, nonnegative s.t. $\varphi_i \nearrow f$, $\psi_i \nearrow g$ pointwise.

Note that $\varphi_i + \psi_i \nearrow f + g$.

Thus,

$$\int (f+g)\,d\mu = \int \left(\lim_{i\to\infty} \varphi_i + \psi_i\right) d\mu$$

↓ Monotone Convergence

$$= \lim_{i\to\infty} \int (\varphi_i + \psi_i)\,d\mu$$

↓ linearity of int. for simple fn

$$= \lim_{i\to\infty} \int \varphi_i\,d\mu + \int \psi_i\,d\mu$$

$$= \lim_{i\to\infty} \int \varphi_i\,d\mu + \lim_{i\to\infty} \int \psi_i\,d\mu$$

↓ Monotone Conv

$$= \int \lim_{i\to\infty} \varphi_i\,d\mu + \int \lim_{i\to\infty} \psi_i\,d\mu$$

$$= \int f\,d\mu + \int g\,d\mu$$

Likewise, $\forall N \in \mathbb{N}$,

$$\int \sum_{n=1}^{N} f_n\,d\mu = \sum_{n=1}^{N} \int f_n\,d\mu.$$

Applying Monotone Convergence Theorem again, as $N \to \infty$,

$$\int \sum_{n=1}^{\infty} f_n \, d\mu \overset{\text{MCT}}{=} \lim_{N \to \infty} \int \sum_{n=1}^{N} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

$\square$

increasing sequence of nonneg meas fns.

One more theorem on interchanging integral and limit...

Thm (Fatou's Lemma):
Given $\{f_n\}_{n=1}^{\infty} : X \to [0, +\infty]$ measureable,

$$\liminf_{n \to \infty} \int f_n \, d\mu \geq \int \liminf_{n \to \infty} f_n \, d\mu$$

Pf: By defn

$$\liminf_{n\to\infty} f_n = \lim_{k\to\infty} \underbrace{\inf_{n\geq k} f_n}_{\text{"}g_k \text{ is increasing"}}$$

By MCT,

$$\lim_{k\to\infty}\int g_k\,d\mu = \int \lim_{k\to\infty} g_k\,d\mu = \int \liminf_{n\to\infty} f_n\,d\mu$$

$$\| \qquad\qquad \|$$

$$\liminf_{k\to\infty}\int g_k\,d\mu$$

For any $k\in\mathbb{N}$, $g_k \leq f_k$, so $\int g_k\,d\mu \leq \int f_k\,d\mu$ and

$$\liminf_{k\to\infty}\int g_k\,d\mu \leq \liminf_{k\to\infty}\int f_k\,d\mu. \quad \square$$

Important Examples of Inequality

$(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{M}_{\lambda^*}, \lambda)$

Ex: ("running away to infinity")

$f_n := 1_{[n, n+1]}$, $\lim_{n\to\infty} f_n = 0$

$$1 = \liminf_{n\to\infty} \int f_n d\lambda > \int \liminf_{n\to\infty} f_n d\lambda = 0$$

Ex: ("going up the spout")

$f_n := n\, 1_{[0, \frac{1}{n}]}$, $\lim_{n\to\infty} f_n(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

$$1 = \liminf_{n\to\infty} \int f_n d\mu > \int \liminf_{n\to\infty} f_n d\lambda = 0$$

In the second example...