Math 201a: Midterm 1

Tuesday, October 20, 2026

Name: _____

Student ID #: _____

Signature: _____

This is a closed-book and closed-note examination. Please show your work in the space provided. You may use scratch paper. You have 1 hour and 15 minutes.

Question	Points	Score
1	10	
2	12	
3	8	
Total	30	

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Question 1 (10 points)

Parts (a) and (b) are not related.

(a) Consider a measure space $(X, \mathcal{M}, \mu)$ and $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$. Recall that

$$\liminf_{i\to\infty} E_i := \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i.$$

Prove that $\liminf_{i\to\infty} \mu(E_i) \geq \mu(\liminf_{i\to\infty} E_i)$.

(b) Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous for every $n \in \mathbb{N}$. Prove that

$$G = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists in } \overline{\mathbb{R}}\}$$

is Borel. You may use the fact that a sequence of real numbers converges if and only if it is Cauchy.

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Question 2 (12 points)

Let X be a nonempty set and suppose μ^* is an outer measure on X. Suppose that, for all $S \subseteq X$ and for all $\epsilon > 0$, there exists a μ^*-measurable set $E \supseteq S$ so that $\mu^*(E) \leq \mu^*(S) + \epsilon$.

(a) Suppose A is not μ^* measurable and consider the σ-algebra $\mathcal{F}$ generated by $\mathcal{M}_{\mu^*}$ and $\{A\}$. Prove that μ^* is not additive on $\mathcal{F}$.

(b) Use part (a) to conclude that $\mathcal{M}_{\mu^*}$ is a maximal σ-algebra on which μ^* can be restricted to be a measure. *(Hint: this is almost immediate from part (a).)*

(c) Consider a measure space $(X, \mathcal{A}, \nu)$. For every $S \subseteq X$, define

$$\mu^*(S) = \inf\{\nu(B) : S \subseteq B \text{ and } B \in \mathcal{A}\}.$$

Use the previous parts to prove that the collection of μ^* measurable sets is a maximal σ-algebra on which μ^* can be restricted to be a measure. *(Hint: you may use the fact, proved on the homework, that the infimum is attained.)*

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Question 3 (8 points)

Suppose μ and ν are finite measures defined on the same measurable space $(X, \mathcal{M})$. Prove that there exists a set $N \in \mathcal{M}$ with the following properties:

(i) $\mu(N) = 0$;

(ii) if $S \in \mathcal{M}$, $S \subseteq X \setminus N$, and $\mu(S) = 0$, then $\nu(S) = 0$.

Hint: among all sets $N \in \mathcal{M}$ with $\mu(N) = 0$, choose the one for which $\nu(N)$ is largest.

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