Math 201A: Midterm 1

Question 1: Solution

(a) Set $A_k = \bigcap_{i=k}^{\infty} E_i$. Then $A_k \in \mathcal{M}$, $A_k \subseteq A_{k+1}$, and $\bigcup_{k=1}^{\infty} A_k = \liminf_{i\to\infty} E_i$. Since $\mu(A_k) \leq \inf_{i\geq k} \mu(E_i)$, continuity from below gives

$$\mu\left(\liminf_{i\to\infty} E_i\right) = \lim_{k\to\infty} \mu(A_k) \leq \lim_{k\to\infty} \inf_{i\geq k} \mu(E_i) = \liminf_{i\to\infty} \mu(E_i).$$

(b) Let $F = \{x \in \mathbb{R} : f_n(x) \text{ converges to a finite real limit}\}$. By the Cauchy criterion, $x \in F$ if and only if, for every $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $|f_i(x) - f_j(x)| \leq 1/m$ for all $i, j \geq N$. Thus

$$F = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{i,j\geq N} \left\{x \in \mathbb{R} : |f_i(x) - f_j(x)| \leq \frac{1}{m}\right\}.$$

The sets inside braces are closed because $f_i - f_j$ is continuous. The set of pairs (i, j) with $i, j \geq N$ is countable, so this expression uses only countable unions and intersections of Borel sets. Therefore F is Borel.

Define the sets where the sequence tends to $+\infty$ or $-\infty$:

$$E_+ = \{x \in \mathbb{R} : f_n(x) \to +\infty\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{x \in \mathbb{R} : f_n(x) > m\},$$

$$E_- = \{x \in \mathbb{R} : f_n(x) \to -\infty\} = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n\geq N} \{x \in \mathbb{R} : f_n(x) < -m\}.$$

Since each f_n is continuous, the innermost sets are open. Thus E_+ and E_- are Borel, being countable intersections and unions of Borel sets.

A sequence converges to a limit in $\overline{\mathbb{R}}$ precisely when it converges to a finite real limit, to $+\infty$, or to $-\infty$. Therefore

$$G = F \cup E_+ \cup E_-.$$

Since F, E_+, and E_- are Borel, G is Borel.

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Question 2: Solution

Write $\mathcal{M} = \mathcal{M}_{\mu^*}$ for the Carathéodory σ-algebra.

(a) Since $A \notin \mathcal{M}$, there is a set $S \subseteq X$ such that

$$\mu^*(S) < \mu^*(S \cap A) + \mu^*(S \cap A^c).$$

In particular, $\mu^*(S) < \infty$, and monotonicity shows that both terms on the right are finite. Choose $\varepsilon > 0$ smaller than the difference between the right and left sides. By the approximation hypothesis, there is $E \in \mathcal{M}$ with $S \subseteq E$ and

$$\mu^*(E) \leq \mu^*(S) + \varepsilon < \mu^*(S \cap A) + \mu^*(S \cap A^c).$$

Since $S \cap A \subseteq E \cap A$ and $S \cap A^c \subseteq E \cap A^c$, monotonicity gives

$$\mu^*(E) < \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

The sets $E \cap A$ and $E \cap A^c$ belong to $\mathcal{F} = \sigma(\mathcal{M} \cup \{A\})$, are disjoint, and have union E. The strict inequality therefore shows that μ^* is not additive on $\mathcal{F}$.

(b) By Carathéodory's theorem, $\mu^*|_{\mathcal{M}}$ is a measure. Suppose that a σ-algebra $\mathcal{G}$ strictly contains $\mathcal{M}$. Choose $A \in \mathcal{G} \setminus \mathcal{M}$. Then $\sigma(\mathcal{M} \cup \{A\}) \subseteq \mathcal{G}$, so part (a) shows that μ^* is not additive on $\mathcal{G}$. Thus $\mathcal{M}$ is maximal under inclusion among σ-algebras on which μ^* is a measure.

(c) Here the induced outer measure is

$$\mu^*(S) = \inf\{\nu(B) : S \subseteq B,\ B \in \mathcal{A}\}.$$

By Homework 3, Question 2(a), the infimum is attained: for every $S \subseteq X$, there is $E \in \mathcal{A}$ with $S \subseteq E$ and $\nu(E) = \mu^*(S)$. By Homework 3, Question 2(c), $E \in \mathcal{M}$ and $\mu^*(E) = \nu(E)$. Hence

$$\mu^*(E) = \mu^*(S) \leq \mu^*(S) + \varepsilon \qquad \text{for every } \varepsilon > 0.$$

The approximation hypothesis holds, even with equality, so part (b) proves the required maximality.

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Question 3: Solution

Let

$$\alpha = \sup\{\nu(A) : A \in \mathcal{M},\ \mu(A) = 0\}.$$

The collection is nonempty because it contains $\emptyset$. Moreover, $0 \le \alpha \le \nu(X) < +\infty$. For each $k \in \mathbb{N}$, choose $N_k \in \mathcal{M}$ such that

$$\mu(N_k) = 0 \qquad \text{and} \qquad \nu(N_k) > \alpha - \frac{1}{k}.$$

Define $N = \bigcup_{k=1}^{\infty} N_k$. Then $N \in \mathcal{M}$, and countable subadditivity gives

$$0 \le \mu(N) \le \sum_{k=1}^{\infty} \mu(N_k) = 0.$$

Thus $\mu(N) = 0$, proving (i). By the definition of α and monotonicity of ν,

$$\alpha \ge \nu(N) \ge \nu(N_k) > \alpha - \frac{1}{k} \qquad \text{for every } k \in \mathbb{N}.$$

Letting $k \to \infty$ gives $\nu(N) = \alpha$.

Now suppose $S \in \mathcal{M}$, $S \subseteq X \setminus N$, and $\mu(S) = 0$. Then $\mu(N \cup S) = 0$. Consequently,

$$\alpha \ge \nu(N \cup S) = \nu(N) + \nu(S) = \alpha + \nu(S).$$

Since $\alpha < +\infty$ and $\nu(S) \ge 0$, this implies $\nu(S) = 0$, proving (ii).