Math 201A: Midterm 2

Solutions

Handwritten pages retain their original question numbers.

Question 1

Let $h_n(x) = (1 + nx^2)/(1 + x^2)^n$ for $x \in [0, 1]$. For every fixed $x > 0$, applying L'Hôpital's rule to the ∞/∞ form gives

$$\lim_{t\to\infty} \frac{1 + tx^2}{(1 + x^2)^t} = \lim_{t\to\infty} \frac{x^2}{\log(1 + x^2)(1 + x^2)^t} = 0.$$

Therefore $h_n(x) \to 0$. At $x = 0$, $h_n(0) = 1$; since $\lambda(\{0\}) = 0$, we have $h_n \to 0$ almost everywhere.

Bernoulli's inequality gives

$$(1 + x^2)^n \geq 1 + nx^2, \qquad 0 \leq h_n(x) \leq 1.$$

The dominating function 1 is integrable on $[0, 1]$. Applying the dominated convergence theorem now yields

$$\boxed{\lim_{n\to\infty} \int_{[0,1]} \frac{1 + nx^2}{(1 + x^2)^n}\, d\lambda(x) = 0.}$$

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(2) First, we prove the hint.

(a) It suffices to show the first result for arbitrary $S \in \mathcal{M}$, since $\mathcal{M}$ is closed under complements.

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By Fatou's lemma and the fact $1_S f_n \to 1_S f$ pointwise and are all nonnegative,

$$\liminf_{n\to\infty} \int 1_S f_n \, d\mu \geq \int \underbrace{\liminf_{n\to\infty} 1_S f_n}_{= 1_S f} \, d\mu$$

By part (a), we have

$$\liminf_{n\to\infty} \int_S f_n \, d\mu \geq \int_S f \, d\mu.$$

Thus, it suffices to show

$$\int_S f \, d\mu \geq \limsup_{n\to\infty} \int_S f_n \, d\mu.$$

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Since $\int f d\mu < +\infty$, we have $\int f_n d\mu < +\infty$ for all but finitely many n.
Thus, by linearity of the integral on $L^1(\mu)$,

$$\limsup_{n\to\infty} \int_S f_n d\mu = \limsup_{n\to\infty} \int (1 - 1_{S^c}) f_n d\mu$$

$$= \limsup_{n\to\infty} \int f_n d\mu - \int_{S^c} f_n d\mu$$

$$\leq \limsup_{n\to\infty} \int f_n d\mu - \liminf_{n\to\infty} \int_{S^c} f_n d\mu$$

$$\leq \int f d\mu - \int_{S^c} f \, d\mu$$

$$= \int_S f d\mu.$$

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(2)

(b) No, the result does not continue to hold.

Consider $f_n := -1_{[-(n+1),-n]} + 1_{[n,n+1]}$

Then $f_n \to 0$ pointwise and $\lim_{n\to\infty} \int f_n d\mu = 0 = \int f d\mu$.

However $\lim_{n\to\infty} \int_{[0,+\infty)} f_n \, d\mu = 1 \neq \int_{[0,+\infty)} f \, d\mu$.

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Question 3

(a) Fix $\epsilon > 0$ and define

$$E_N = \bigcup_{n \geq N} \{x : |f_n(x) - f(x)| \geq \epsilon\}.$$

The sets E_N are measurable and decrease as N increases. Wherever $f_n(x) \to f(x)$, the inequality $|f_n(x) - f(x)| < \epsilon$ holds for all sufficiently large n, so $x \notin \bigcap_{N=1}^{\infty} E_N$. Thus $\mu(\bigcap_{N=1}^{\infty} E_N) = 0$.

Since $\mu(E_1) \leq \mu(X) < \infty$, continuity from above gives

$$\mu(E_N) \longrightarrow \mu\left(\bigcap_{N=1}^{\infty} E_N\right) = 0.$$

Finally,

$$\mu\big(\{|f_N - f| \geq \epsilon\}\big) \leq \mu(E_N) \longrightarrow 0,$$

which proves convergence in measure.

(b) Take $X = \mathbb{R}$ with Lebesgue measure λ, and let $f_n = \mathbf{1}_{[n,n+1]}$ and $f = 0$. For every fixed $x \in \mathbb{R}$, eventually $x \notin [n, n+1]$, so $f_n(x) \to 0$ everywhere. However,

$$\lambda\big(\{x : |f_n(x) - f(x)| \geq 1/2\}\big) = \lambda([n, n+1]) = 1$$

for every n. Thus f_n does not converge to 0 in measure, even though it converges everywhere and $\lambda(\mathbb{R}) = +\infty$.

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