Math 201A: Practice Midterm 1

Not to be turned in.

Question 1

(i) Prove that $\mathcal{B}_{\bar{\mathbb{R}}}$ is generated by sets of the form $(a, +\infty]$ for $a \in \mathbb{R}$.

(ii) Prove that $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$.

Hint: You may use the fact that the intervals (a, b), $[-\infty, b)$, and $(a, +\infty]$, with $a, b \in \mathbb{Q}$ and $a < b$ for the first type, form a countable base for the topology of $\bar{\mathbb{R}}$. Consequently, every open subset of $\bar{\mathbb{R}}$ is a countable union of these intervals.

Question 2

Consider a measure space $(X, \mathcal{M}, \mu)$ and $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$. Recall that $\limsup_{i\to+\infty} E_i := \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} E_i$.

(a) If $\sum_{i=1}^{\infty} \mu(E_i) < +\infty$, prove that $\mu(\limsup_{i\to+\infty} E_i) = 0$.

(b) Suppose $\mu(X) = 1$. If $\sum_{i=1}^{\infty} \mu(E_i) = +\infty$ and $\{E_i\}_{i=1}^{\infty}$ satisfy

$$\mu\left(\bigcap_{i=n}^{N} E_i^c\right) = \prod_{i=n}^{N} \mu(E_i^c), \qquad \text{for all } n, N \in \mathbb{N},\ n \leq N,$$

prove that $\mu(\limsup_{i\to+\infty} E_i) = 1$.

Hint: Use the estimate $\log(x) \leq x - 1$ for $x \in (0, 1]$ to obtain $\sum_{i=n}^{N} \log(\mu(E_i^c)) \leq -\sum_{i=n}^{N} \mu(E_i)$.

Question 3

Let μ be a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Recall from Homework 2, Question 10 that, for any $T \in \mathcal{B}_{\mathbb{R}}$, the restriction of μ to T, defined by $\mu_T(S) := \mu(S \cap T)$ for $S \in \mathcal{B}_{\mathbb{R}}$, is a measure on $\mathcal{B}_{\mathbb{R}}$.

Now, suppose $\mu(B) < +\infty$ for all balls $B \subseteq \mathbb{R}$. (This is called a *locally finite measure*.) Prove that, for any $S \in \mathcal{B}_{\mathbb{R}}$ and for all $\epsilon > 0$, there exists an open set U satisfying $S \subseteq U$ so that $\mu(U \setminus S) \leq \epsilon$. Hint: consider the restriction to $T = B$.

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