Question 1

Practice Midterm 1

MATH 201A: HOMEWORK 3, QUESTION 10

Solution

(i) Let

$$\mathcal{F} = \{(a, +\infty] : a \in \mathbb{R}\}.$$

Each member of $\mathcal{F}$ is open in $\bar{\mathbb{R}}$, so

$$\mathcal{M}(\mathcal{F}) \subseteq \mathcal{B}_{\bar{\mathbb{R}}}.$$

For the reverse inclusion, observe that, for every $b \in \mathbb{R}$,

$$[-\infty, b) = \bigcup_{n=1}^{\infty} \left(\bar{\mathbb{R}} \setminus (b - 1/n, +\infty]\right) \in \mathcal{M}(\mathcal{F}).$$

Consequently, for $a < b$,

$$(a, b) = (a, +\infty] \cap [-\infty, b) \in \mathcal{M}(\mathcal{F}).$$

The intervals (a, b), $[-\infty, b)$, and $(a, +\infty]$, with finite rational endpoints, form a countable basis for the topology of $\bar{\mathbb{R}}$. Every open set is therefore a countable union of members of $\mathcal{M}(\mathcal{F})$. Hence

$$\mathcal{B}_{\bar{\mathbb{R}}} = \mathcal{M}(\mathcal{F}).$$

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(ii) Define

$$\mathcal{A} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

This is a σ-algebra on $\bar{\mathbb{R}}$: intersections with $\mathbb{R}$ preserve countable unions, and

$$(\bar{\mathbb{R}} \setminus E) \cap \mathbb{R} = \mathbb{R} \setminus (E \cap \mathbb{R}).$$

Moreover, $\mathcal{F} \subseteq \mathcal{A}$, since

$$(a, +\infty] \cap \mathbb{R} = (a, +\infty) \in \mathcal{B}_{\mathbb{R}}.$$

Part (i) thus gives $\mathcal{B}_{\bar{\mathbb{R}}} \subseteq \mathcal{A}$.

For the reverse inclusion, first note that

$$\{+\infty\} = \bigcap_{n=1}^{\infty} (n, +\infty], \qquad \{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n)$$

are Borel subsets of $\bar{\mathbb{R}}$. Therefore $\mathbb{R}$ and every ray

$$(a, +\infty) = (a, +\infty] \setminus \{+\infty\}$$

also belong to $\mathcal{B}_{\bar{\mathbb{R}}}$.

By HW2, Question 4(iii), these rays generate $\mathcal{B}_{\mathbb{R}}$. Thus, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\bar{\mathbb{R}}}$.

Now let $E \in \mathcal{A}$. Then

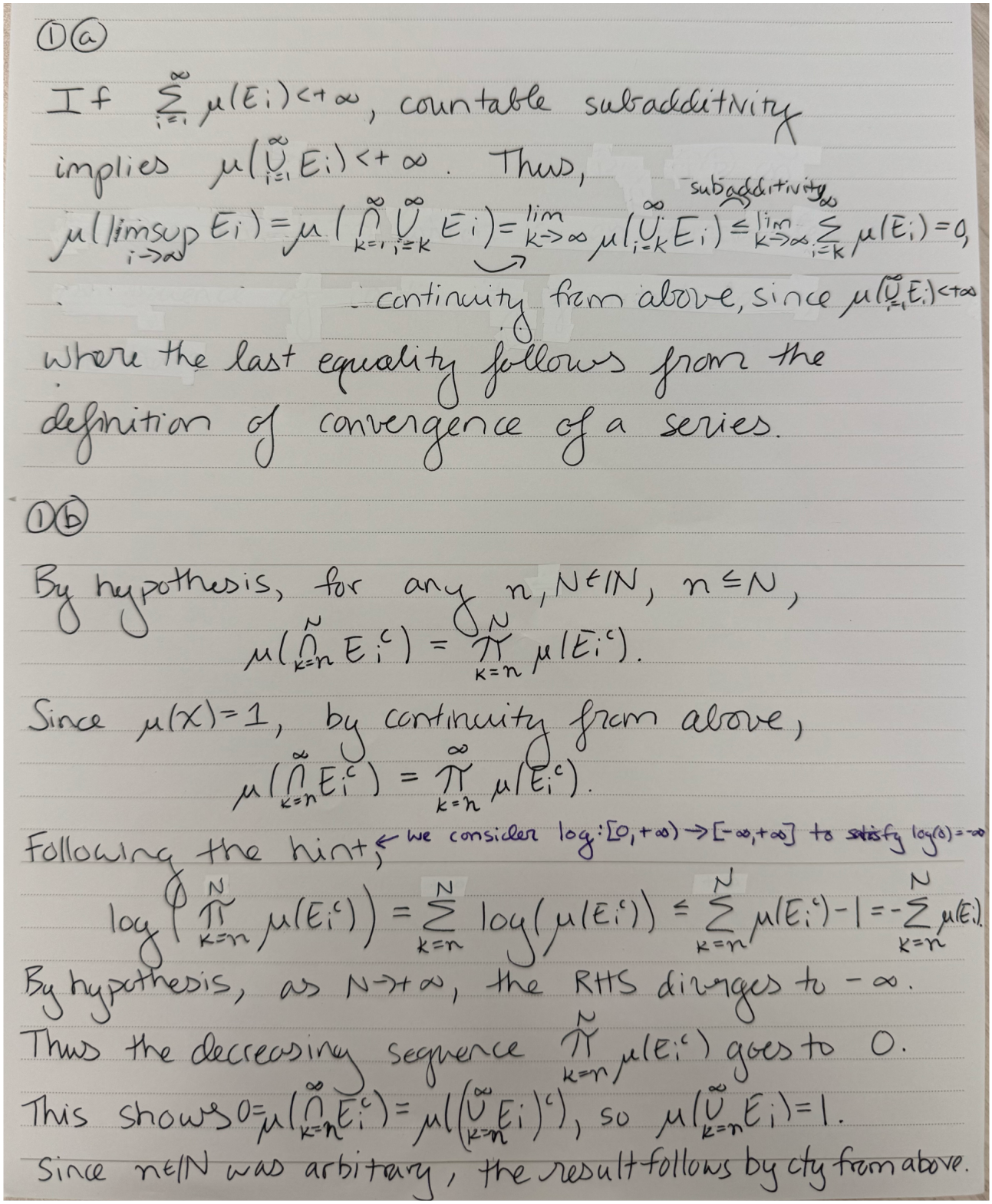
$$E = (E \cap \mathbb{R}) \cup (E \cap \{-\infty, +\infty\}).$$

The first set belongs to $\mathcal{B}_{\bar{\mathbb{R}}}$ by the preceding paragraph. The second is a union of at most two Borel singletons. Thus $E \in \mathcal{B}_{\bar{\mathbb{R}}}$, proving $\mathcal{A} \subseteq \mathcal{B}_{\bar{\mathbb{R}}}$. Therefore

$$\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

Question 2

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① (a)

If $\sum_{i=1}^{\infty} \mu(E_i) < +\infty$, countable subadditivity implies $\mu(\bigcup_{i=1}^{\infty} E_i) < +\infty$. Thus,

$$\mu(\limsup_{i\to\infty} E_i) = \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} E_i\right) = \lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} E_i\right) \leq \lim_{k\to\infty} \sum_{i=k}^{\infty} \mu(E_i) = 0,$$

(subadditivity; continuity from above, since $\mu(\bigcup_{i=1}^{\infty} E_i) < +\infty$)

where the last equality follows from the definition of convergence of a series.

① (b)

By hypothesis, for any $n, N \in \mathbb{N}$, $n \leq N$,

$$\mu\left(\bigcap_{k=n}^{N} E_i^c\right) = \prod_{k=n}^{N} \mu(E_i^c).$$

Since $\mu(X) = 1$, by continuity from above,

$$\mu\left(\bigcap_{k=n}^{\infty} E_i^c\right) = \prod_{k=n}^{\infty} \mu(E_i^c).$$

Following the hint; ← we consider $\log : [0, +\infty) \to [-\infty, +\infty]$ to satisfy $\log(0) = -\infty$

$$\log\left(\prod_{k=n}^{N} \mu(E_i^c)\right) = \sum_{k=n}^{N} \log(\mu(E_i^c)) \leq \sum_{k=n}^{N} \mu(E_i^c) - 1 = -\sum_{k=n}^{N} \mu(E_i).$$

By hypothesis, as $N \to +\infty$, the RHS diverges to $-\infty$.

Thus the decreasing sequence $\prod_{k=n}^{N} \mu(E_i^c)$ goes to 0.

This shows $0 = \mu\left(\bigcap_{k=n}^{\infty} E_i^c\right) = \mu\left(\left(\bigcup_{k=n}^{\infty} E_i\right)^c\right)$, so $\mu\left(\bigcup_{k=n}^{\infty} E_i\right) = 1$.

Since $n \in \mathbb{N}$ was arbitary, the result follows by cty from above.

Question 3

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(b) As shown in class, all finite Borel measures are Lebesgue-Stieltjes measures, hence outer regular. Let $B_i := (-i, i)$, $i \in \mathbb{N}$. Then $\mu(B_i) < +\infty$, so $\mu_i := \mu_{B_i}$ from part (a) is a

finite Borel measure. Fix $S \in \mathcal{B}_{\mathbb{R}}$ and $\varepsilon > 0$. Then $\exists U_i$ open so that $S \subseteq U_i$ and

$$\mu_i(U_i) < \mu_i(S) + \frac{\varepsilon}{2^i}.$$

Thus

$$\mu(S^c \cap U_i \cap B_i) = \mu_i(U_i \setminus S) \leq \frac{\varepsilon}{2^i}$$

Let $V = \bigcup_{i=1}^{\infty}(B_i \cap U_i)$. Then V is open and

$$S \subseteq \bigcup_{i=1}^{\infty} B_i,\ S \subseteq U_i\ \forall i \Rightarrow S \subseteq \bigcup_{i=1}^{\infty} B_i \cap U_i = V.$$

Therefore, by countable subadditivity,

$$\mu\left(\bigcup_{i=1}^{\infty} B_i \cap (U_i \setminus S)\right) \leq \sum_{i=1}^{\infty} \mu(B_i \cap (U_i \setminus S)) \leq \varepsilon$$

$$\mu\left(\bigcup_{i=1}^{\infty} B_i \cap U_i \cap S^c\right)$$

$$\mu(V \cap S^c)$$

$$\mu(V \setminus S).$$

Right idea but lacking details: 7/10

Outer regular: +3

Set difference: +1