Math 201A: Practice Midterm 2

Not to be turned in.

Question 1

Compute the following limit and justify your calculations. You do not need to explain why the functions are measurable. Explain where you use standard Calculus facts about the Riemann integral and where you use the equivalence of the Riemann and Lebesgue integral.

$$\lim_{n\to+\infty} \int_0^\infty (1+(x/n))^{-n} \sin(x/n) d\lambda(x).$$

Question 2

Consider a measure space $(X, \mathcal{M}, \mu)$ with $\mu(X) < +\infty$.

In this question, you will prove that convergence in measure is *metrizable*: there exists a metric on the space of measurable functions (up to almost everywhere equivalence) so that convergence in this metric is equivalent to convergence in measure.

(a) Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(s) = s/(1+s)$. Prove that $\phi(s)$ is nondecreasing, $\phi(s+t) \leq \phi(s) + \phi(t)$ and $\phi(s) = 0 \iff s = 0$.

(b) Given $f, g : X \to \mathbb{R}$ measurable, define

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|} d\mu.$$

Prove that ρ is a metric on the space of measurable functions, if we identify functions that are equal almost everywhere.

(c) Given $f_n, f : X \to \mathbb{R}$ measurable, show that

$$\rho(f_n, f) \to 0 \iff f_n \to f \text{ in measure.}$$

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Question 3

In this problem, you may use the following fact without proof:

$$||a+b|-|a|-|b|| \leq 2|b| \ , \quad \forall a, b \in \mathbb{R}.$$

Consider a measure space $(X, \mathcal{M}, \mu)$ and $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ satisfying $f_n \to f$ μ-almost everywhere.

(a) Suppose there exists $M > 0$ so that $\|f_n\|_{L^1(\mu)} \leq M$ $\forall n \in \mathbb{N}$. Prove that

$$f \in L^1(\mu) \quad \text{and} \quad \lim_{n \to +\infty} \int (|f_n| - |f_n - f|)\, d\mu = \int |f| d\mu.$$

Hint: take $a = f_n - f$ and $b = f$.

(b) Suppose that $f \in L^1(\mu)$ and $\lim_{n \to +\infty} \|f_n\|_{L^1(\mu)} = \|f\|_{L^1(\mu)}$. Prove that

$$\lim_{n \to +\infty} \|f_n - f\|_{L^1(\mu)} = 0.$$

Extra Practice Problems

Question 4

Suppose $(X, \mathcal{M}, \mu)$ is a **finite** measure space and $f : X \to [0, +\infty)$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable. Define the floor function $\lfloor \cdot \rfloor : [0, +\infty) \to \mathbb{R}$ by

$$\lfloor y \rfloor = \begin{cases} \sup\{n \in \mathbb{N} : n \leq y\} & \text{for } y \geq 1, \\ 0 & \text{for } y < 1. \end{cases}$$

(a) Prove that $x \mapsto \lfloor f(x) \rfloor$ is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable.

(b) Prove that

$$\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\}) = \int \lfloor f \rfloor d\mu.$$

(c) Prove that $f \in L^1(\mu)$ if and only if

$$\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\}) < +\infty.$$

Question 5

Suppose $\{f_n\}_{n=1}^{\infty} \subseteq L^1(\mu)$ and $f_n \to f$ uniformly.

(a) If $\mu(X) < +\infty$, prove that $f \in L^1(\mu)$ and $\int f_n d\mu \to \int f d\mu$.

(b) If $\mu(X) = +\infty$, give a counterexample to show the result of part (a) fails.

(c) Suppose μ is the counting measure on $\mathbb{N}$ endowed with the σ-algebra $2^{\mathbb{N}}$. Prove that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

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