Math 201A: Practice Midterm 2

Solutions

Handwritten pages retain their original question numbers.

Question 1

(a)

To see that $n \mapsto (1+\frac{x}{n})^n$ is increasing, it suffices to show $y \mapsto n\log(1+\frac{x}{n})$ is increasing for $x, y > 0$. This follows since it is differentiable and

$$\frac{\partial}{\partial y} y\log\left(1+\frac{x}{y}\right) = \log\left(1+\frac{x}{y}\right) + \frac{y}{1+\frac{x}{y}}$$

$$= \log\left(1+\frac{x}{y}\right) + \frac{y^2}{x+y} \geq 0.$$

4pts

Therefore, $\forall n \geq 2$, $\left(1+\frac{x}{n}\right)^{-n} \leq \left(1+\frac{x}{2}\right)^{-2}$.

Consequently, $g(x) = (1+\frac{x}{2})^{-2}$ is a dominating function for the sequence. Furthermore,

Monotone Convergence Theorem

$$\int_{(0,\infty)} g(x)\,dx = \lim_{R\to\infty} \int_{(0,R)} g(x)\,dx$$

Equivalence of Riemann and Lebesgue

$$= \lim_{R\to\infty} \int_0^R g(x)\,dx$$

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$$= \lim_{R\to\infty} \left[(2)\left(1+\frac{x}{2}\right)^{-1}\right]_0^R$$

$$= 2$$

Thus $g \in L^1(\lambda)$. 4pts

By the dominated convergence theorem, 4pts

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) d\lambda(x)$$

$$= \int_0^\infty \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) d\lambda(x)$$

$$= \int_0^\infty e^{-x} \cdot 0 \, d\lambda(x)$$

$$= 0$$

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(1)

(a) $\phi'(s) = \frac{1+s-s}{(1+s)^2} = \frac{1}{(1+s)^2} \geq 0 \Rightarrow \phi$ is nondecreasing

$\phi(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = \phi(s)+\phi(t)$

$\phi(s) = 0 \Leftrightarrow \frac{s}{1+s} = 0 \Leftrightarrow s = 0$, since $s \geq 0 \Leftrightarrow 1+s > 0$.

(b) $\rho(f,g) = 0$

$\Leftrightarrow \underbrace{\frac{|f-g|}{1+|f-g|}}_{\psi :=} = 0$ a.e., since ψ is a nonneg. meas fn.

$\Leftrightarrow \phi(|f(x)-g(x)|) = 0$ a.e.

$\Leftrightarrow |f(x)-g(x)| = 0$ a.e.

$\Leftrightarrow f = g$ a.e.

$\rho(f,g) = \int \phi(|f-g|)\,d\mu = \int \phi(|f-h+h-g|)\,d\mu$

(ϕ nondecreasing)

$\leq \int \phi(|f-h| + |h-g|)\,d\mu$

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$$\leq \int \phi(|f-h|) + \phi(|h-g|)\, d\mu \quad \text{(a)}$$

$$\leq \int \phi(|f-h|)\, d\mu + \int \phi(|h-g|)\, d\mu \quad \text{Beppo Levi}$$

$$= \rho(f,h) + \rho(h,g)$$

(c) Note that

$$\int \phi(|f_n - f|)\, d\mu \geq \int_{\{x : |f_n(x) - f(x)| \geq \varepsilon\}} \phi(|f_n - f|)\, d\mu \geq \int_{\{x : |f_n(x) - f(x)| \geq \varepsilon\}} \phi(\varepsilon)\, d\mu \quad \phi \text{ nondecreasing}$$

$$\geq \frac{\varepsilon}{1+\varepsilon}\, \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}).$$

Thus, $\rho(f_n, f) \to 0$ implies $f_n \to f$ in measure.

Conversely, for all $\varepsilon > 0$.

$$\int \phi(|f_n - f|)\, d\mu \leq \int_{\{x : |f_n(x) - f(x)| \geq \varepsilon\}} \phi(|f_n - f|)\, d\mu + \int_{\{x : |f_n(x) - f(x)| < \varepsilon\}} \phi(|f_n - f|)\, d\mu$$

$$\leq \int_{\{x : |f_n(x) - f(x)| \geq \varepsilon\}} 1\, d\mu + \int_{\{x : |f_n(x) - f(x)| < \varepsilon\}} \phi(\varepsilon)\, d\mu \quad \phi \leq 1,\ \phi \text{ increasing}$$

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$$\leq \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) + \frac{\varepsilon}{1-\varepsilon}\mu(X)$$

Thus,

$$\limsup_{n\to\infty} \int \phi(|f_n - f|)\, d\mu \leq \frac{\varepsilon}{1-\varepsilon}\mu(X).$$

Since $\varepsilon > 0$ was arbitrary and $\mu(X) < +\infty$, this gives the result.

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Question 3

We work with a measurable representative of the almost-everywhere limit f.

(a) Since $f_n \to f$ μ-almost everywhere, $\liminf_{n\to\infty} |f_n| = \lim_{n\to\infty} |f_n| = |f|$ μ-almost everywhere, so their integrals coincide because changing a function on a null set does not change its integral.

By Fatou's lemma,

$$\int |f|\, d\mu \le \liminf_{n\to\infty} \int |f_n|\, d\mu \le M < \infty,$$

so $f \in L^1(\mu)$. Define

$$h_n = |f_n| - |f_n - f| - |f|.$$

Then $h_n \to 0$ almost everywhere. The given inequality, with $a = f_n - f$ and $b = f$, gives $|h_n| \le 2|f|$ almost everywhere. Since $2|f| \in L^1(\mu)$, the dominated convergence theorem yields $\int h_n\, d\mu \to 0$. Therefore,

$$\int \big(|f_n| - |f_n - f|\big)\, d\mu = \int h_n\, d\mu + \int |f|\, d\mu \longrightarrow \int |f|\, d\mu.$$

(b) Since $\|f_n\|_{L^1(\mu)} \to \|f\|_{L^1(\mu)} < \infty$, the sequence of norms is bounded, so part (a) applies. By linearity of the integral for $L^1(\mu)$ functions,

$$\|f_n - f\|_{L^1(\mu)} = \|f_n\|_{L^1(\mu)} - \int \big(|f_n| - |f_n - f|\big)\, d\mu \longrightarrow \|f\|_{L^1(\mu)} - \|f\|_{L^1(\mu)} = 0.$$

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Question 4

(2)

(a) Note that $\sum_{n=1}^{N} 1_{[n,+\infty)}(y)$ is a simple fn for all $N \in \mathbb{N}$, hence measurable. Since it is increasing in N, the limit as $N \to +\infty$ exists, so $\lfloor y \rfloor$ is $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$-measurable. Thus the composition $x \mapsto \lfloor f(x) \rfloor$ is measurable.

compatible σ-algebras −0.5

(b)

$$\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\}) = \sum_{n=1}^{\infty} \int 1_{\{x : f(x) \geq n\}} d\mu \quad (+1) \quad (+1)$$

Beppo-Levi

$$= \int \sum_{n=1}^{\infty} 1_{\{x : f(x) \geq n\}} d\mu \quad (+1)$$

part (a)

$$= \int \lfloor f \rfloor \, d\mu \quad (+1)$$

(c)

By definition $\lfloor f \rfloor \leq f \leq \lfloor f \rfloor + 1$. Thus, by monotonicity of the integral of nonnegative measurable functions

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$$\int \lfloor f \rfloor d\mu \leq \int f d\mu \leq \int \lfloor f \rfloor + 1 \, d\mu \leq \int \lfloor f \rfloor d\mu + \mu(X).$$

Since $\mu(X) < +\infty$, this shows $\int f d\mu < +\infty$ iff $\int \lfloor f \rfloor d\mu < +\infty$. Since f is nonnegative, applying part ©, we obtain $f \in L^1(\mu)$ iff

$$\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\}) < +\infty.$$

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Question 5

(a) The pointwise limit f is measurable. Uniform convergence allows us to choose N so that $|f - f_N| \leq 1$ everywhere. Hence

$$|f| \leq |f_N| + 1, \qquad \int |f|\, d\mu \leq \int |f_N|\, d\mu + \mu(X) < \infty.$$

Thus $f \in L^1(\mu)$. Given $\epsilon > 0$, uniform convergence gives $|f_n - f| < \epsilon$ everywhere for all sufficiently large n. Therefore, by linearity of the integral for $L^1(\mu)$ functions,

$$\left| \int f_n\, d\mu - \int f\, d\mu \right| \leq \int |f_n - f|\, d\mu \leq \epsilon \mu(X).$$

This proves convergence of the integrals.

(b) On $\mathbb{R}$ with Lebesgue measure, let $f_n = n^{-1} 1_{[0,n]}$ and $f = 0$. Then $f_n \in L^1(\lambda)$ and $\sup_x |f_n(x)| = 1/n \to 0$, but

$$\int f_n\, d\lambda = 1 \quad \text{for every } n, \qquad \int f\, d\lambda = 0.$$

Thus the integral conclusion can fail when the space has infinite measure.

(c) If $f_n \to f$ uniformly, then for every $\epsilon > 0$ the set $A_{n,\epsilon} = \{k \in \mathbb{N} : |f_n(k) - f(k)| \geq \epsilon\}$ is empty for all sufficiently large n. Hence $f_n \to f$ in measure.

Conversely, if $f_n \to f$ in counting measure, then $\mu(A_{n,\epsilon}) \to 0$. Every nonempty set has counting measure at least 1, so eventually $A_{n,\epsilon} = \varnothing$. Thus $|f_n(k) - f(k)| < \epsilon$ for every k once n is sufficiently large. This is uniform convergence.

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