Lecture 16

Recall:

Thm $(W_2, \mathcal{P}_2(\mathbb{R}^d))$ is a metric space.

Lemma: $\mu_n \to \mu$ narrowly iff $\int f d\mu_n \to \int f d\mu$ $\forall f \in C_c^\infty(\mathbb{R}^d)$.

Thm: Given $\mu_n, \mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\lim_{n\to\infty} W_2(\mu_n, \mu) = 0 \iff \begin{cases} \mu_n \to \mu \text{ narrowly} \\ m_2(\mu_n) \to m_2(\mu) \end{cases}$$

Lemma: Suppose $\exists X \subset\subset \mathbb{R}^d$ s.t. $\forall n$, $\mu_n(X^c) = \mu(X^c) = 0$. Then $\mu_n \to \mu$ narrowly $\iff$ $W_2(\mu_n, \mu) \to 0$.

Pf:

Last time, $\mu_n \xrightarrow{W_2} \mu \Rightarrow \mu_n \xrightarrow{\text{narrowly}} \mu$, $m_2(\mu_n) \to m_2(\mu)$.

Remains to show $\mu_n \xrightarrow{\text{narrowly}} \mu \Rightarrow \mu_n \xrightarrow{W_2} \mu$.

By exercise, it suffices to show $W_1(\mu_n, \mu) \to 0$

Choose subseq s.t. $\lim_{k\to\infty} W_1(\mu_{n_k}, \mu) = \limsup_{n\to\infty} W_1(\mu_n, \mu)$

By duality, $\exists \varphi_{n_k} \in C(X)$, $\|\varphi_{n_k}\|_{Lip} \le 1$ s.t.

$$W_1(\mu_{n_k}, \mu) = \int \varphi_{n_k} d\mu_{n_k} - \int \varphi_{n_k} d\mu .$$

WLOG (shifting φ_{n_k} by const doesn't affect RHS), we may assume $\{\varphi_{n_k}\}$ unif bdd.

By Arzela-Áscoli, $\exists$ further subsequence s.t. $\varphi_{n_k} \to \varphi \in C(X)$ unif on X.

$$\limsup_{n\to\infty} W_1(\mu_n, \mu) = \lim_{k\to\infty} W_1(\mu_{n_k}, \mu)$$

$$= \lim_{k\to\infty} \int (\varphi_{n_k} - \varphi)\, d(\mu_{n_k} - \mu) + \int \varphi\, d(\mu_{n_k} - \mu)$$

$$= 0 \qquad \square$$

Back to proof of theorem.

Pf: Last time "$\Rightarrow$".

Now, "$\Leftarrow$".

We will do this by restricting our measures to cpt set and applying lem.

For $R>0$, define

$$\pi_R(x) = \begin{cases} x & \text{if } |x|\leq R \\ R\frac{x}{|x|} & \text{if } |x|>R \end{cases}$$

Since $\pi_R \in C_b(\mathbb{R}^d, \mathbb{R}^d)$, $\mu_n \xrightarrow{\text{narrowly}} \mu$ implies $\pi_R\#\mu_n \xrightarrow{\text{narrowly}} \pi_R\#\mu$.

Furthermore, $\forall\, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,
$(\pi_R\#\nu)(\overline{B_R(0)}^c) = \nu(\pi_R^{-1}(\overline{B_R(0)}^c)) = \nu(\emptyset) = 0.$

So $\pi_R\#\mu_n, \pi_R\#\mu$ concentrated on cpt set. Lemma ensures

$$\pi_R\#\mu_n \xrightarrow{W_2} \pi_R\#\mu.$$

Now, we seek to send $R \nearrow +\infty$.

Note that, $\forall\, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ $\quad (id\times\pi_R)\#\nu \in \Gamma(\nu, \pi_R\#\nu)$

$$W_2^2(\nu, \pi_R\#\nu) \leq \int |\pi_R(x) - x|^2 d\nu$$

$$= \int_{\overline{B}_R^c} \left|R\frac{x}{|x|} - x\right|^2 d\nu$$

$$= \int_{\overline{B}_R^c} |x|^2 - 2R|x| + R^2 \, d\nu$$

$$\leq \int_{\overline{B}_R^c} |x|^2 - R^2 \, d\nu$$

$$= \int_{\overline{B}_R^c} |x|^2 - |\pi_R(x)|^2 \, d\nu \quad // \int_{\mathbb{R}^d} |x|^2 - |\pi_R(x)|^2 \, d\nu$$

$$\leq \int_{\overline{B}_R^c} |x|^2 \, d\nu$$

Now we combine this estimate with the following facts:

- $|\pi_R(x)|^2 \in C_b(\mathbb{R}^d) \Rightarrow \int |\pi_R(x)|^2 \, d\mu_n \to \int |\pi_R(x)|^2 \, d\mu$
- $M_2(\mu_n) \to M_2(\mu) \Rightarrow \int |x|^2 \, d\mu_n \to \int |x|^2 \, d\mu$

Thus,

$$\limsup_{n\to\infty} W_2^2(\mu_n, \pi_R \# \mu_n)$$

$$\leq \limsup_{n\to\infty} \int_{\mathbb{R}^d} |x|^2 - |\pi_R(x)|^2 \, d\mu_n = \int_{\mathbb{R}^d} |x|^2 - |\pi_R(x)|^2 \, d\mu$$

$$\leq \int_{\overline{B}_R^c} |x|^2 d\mu$$

Finally, since $\int |x|^2 d\mu(x) < +\infty$, $\forall \varepsilon > 0$, $\exists R_\varepsilon$ s.t. $\int_{\overline{B}_{R_\varepsilon}^c} |x|^2 d\mu(x) < \varepsilon$.

Thus, $\forall \varepsilon > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} W_2(\mu_n, \mu) &\leq \limsup_{n\to\infty} W_2(\pi_{R_\varepsilon} \# \mu_n, \mu_n) \\
&\quad + \limsup_{n\to\infty} W_2(\pi_{R_\varepsilon} \# \mu, \mu) \\
&\quad + \limsup_{n\to\infty} W_2(\pi_{R_\varepsilon} \# \mu_n, \pi_{R_\varepsilon} \# \mu) \\
&\leq \varepsilon + \varepsilon + 0 \qquad \square
\end{aligned}$$

- Final topic: connect OT to PDE

- characterize absolutely continuous curves in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ as solutions of the continuity equation.

- dynamic characterization of W_2 (Benamou - Brenier)

The Continuity Equation (CE)

<u>Def</u>: (Weak solution of continuity eqn):
Given $v : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ meas, $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$, $\mu : (0,T) \to \mathcal{M}(\mathbb{R}^d)$ is a weak solution of the continuity equation

$$\begin{cases} \partial_t \mu + \nabla \cdot (\mu v) = 0 \\ \mu(0) = \mu_0 \end{cases}$$

if

- $\mu(t)$ is narrowly cts, $\lim_{t \to 0^+} \mu(t) = \mu_0$ narrowly
- the PDE holds in distribution, that is

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi(x,t) + \nabla \varphi(x,t) \cdot v(x,t) \right) d\mu_t(x)\, dt = 0$$

$$\forall\, \varphi \in C_c^\infty(\mathbb{R}^d \times (0,T)).$$

The above definition is the <u>Eulerian</u> perspective on solutions of (CE).
There is also a <u>Lagrangian</u> perspective.

Def: (solution of ODE): Given $v: \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ meas and $x_0 \in \mathbb{R}^d$, $x: (0,T) \to \mathbb{R}^d$ is a solution of

$$\begin{cases} x'(t) = v(x(t), t) \\ x(0) = x_0 \end{cases}$$

if

- $x(t)$ is locally abs cts, $\lim_{t \to 0^+} x(t) = x_0$.
- the ODE holds in the integral sense, i.e.

$$x(t) = x_0 + \int_0^t v(x(s), s)\, ds, \quad \forall t \in [0,T].$$

A classical result in ODE is...

Thm (Cauchy Lipschitz):
Suppose

- $v: \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ is meas
- $\|v(\cdot, t)\|_{Lip} < +\infty \ \forall t \in (0,T)$, $\|v(\cdot,t)\|_{Lip} \in L^1_{loc}(0,T)$
- $\exists\ C > 0$ s.t. $|v(x,t)| \leq C(1 + |x|)$

Then, $\forall x_0 \in \mathbb{R}^d$, $\exists !$ solution of ODE and

- $|x(t)| \leq f(t)(1+|x|)$, for $f(t):(0,T) \to [0,+\infty)$ dep. on C
- $|x(t)-y(t)| \leq e^{\int_0^t \|v(\cdot,s)\|_{Lip} ds} |x_0 - y_0|$

Fix $v(x,t)$. Suppose that a solution of (ODE) exists $\forall x_0 \in \mathbb{R}^d$. In this case, we may consider the flow map induced by v,

$X_t(y) = X(y,t) = x(t)$, where $x(t)$ is soln w/ $x(0)=y$.

Pushing forward a measure by this flow map gives us a soln of (CE) w/ velocity v. This provides the Lagrangian perspective.

Prop: Fix $v(x,t)$. Suppose that solns of (ODE) exist globally in time $\forall x_0 \in \mathbb{R}^d$. Let X_t be the corresponding flow map.

Then for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, define $\mu_t = X_t \# \mu_0$.

Then,

- $\mu_t \in \mathcal{P}_2(\mathbb{R}^d) \quad \forall\, t > 0$
- If $\int_0^T \int_{\mathbb{R}^d} |v(x,t)|\, d\mu_t(x)\, dt < +\infty$,

  then (μ, v) is a soln of (CE).

Pf:

Since

$$\left\{ \sum_{i=1}^{N} \alpha_i(t) f(x) : \alpha_i \in C_c^\infty(0,T),\ f_i \in C_c^\infty(\mathbb{R}^d),\ N \in \mathbb{N} \right\}$$

is dense $C_c^\infty(\mathbb{R}^d \times (0,T))$, it suffices to show that

$$0 = \int_0^T \int_{\mathbb{R}^d} \left(\partial_t \alpha(t) f(x) + \alpha(t) \nabla f(x) \cdot v(x,t) \right) d\mu_t(x)\, dt$$

$$= \int_0^T \partial_t \alpha(t) \left(\int_{\mathbb{R}^d} f(x)\, d\mu_t(x) \right) + \alpha(t) \left(\int_{\mathbb{R}^d} \nabla f(x) v(x,t)\, d\mu_t(x) \right) dt$$

for all $\alpha \in C_c^\infty(0,T)$, $f \in C_c^\infty(\mathbb{R}^d)$.

It suffices to show

$t \mapsto \int_{\mathbb{R}^d} f(x)\, d\mu_t(x)$ is abs cts and

its time derivative is $\int_{\mathbb{R}^d} \nabla f(x) \cdot v(x,t)\, d\mu_t(x)$

To see this, note that for $s<t$,

$$\int_{\mathbb{R}^d} f\, d\mu_t - \int f\, d\mu_s = \int_{\mathbb{R}^d} (f \circ X_t - f \circ X_s)\, d\mu_0$$
$$= \int_{\mathbb{R}^d} \int_s^t \frac{d}{dr} f \circ X_r\, dr\, d\mu_0$$
$$= \int_{\mathbb{R}^d} \int_s^t \nabla f(X_r(y))\, v(X_r(y), r)\, dr\, d\mu_0(y)$$
$$= \int_s^t \int_{\mathbb{R}^d} \nabla f(x)\, v(x,r)\, d\mu_r(x)\, dr. \qquad \square$$

We will now use this Lagrangian perspective to connect Wasserstein geodesics to (CE).