Lecture 17

Announcements:

- Revision due by March 4^{th}

Recall:

Def: (Weak solution of continuity eqn): Given $v : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ meas, $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$, $\mu : (0,T) \to \mathcal{M}(\mathbb{R}^d)$ is a weak solution of the continuity equation

$$\begin{cases} \partial_t \mu + \nabla \cdot (\mu v) = 0 \\ \mu(0) = \mu_0 \end{cases}$$

if

- $\mu(t)$ is narrowly cts, $\lim_{t \to 0^+} \mu(t) = \mu_0$ narrowly
- the PDE holds in distribution, that is

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi(x,t) + \nabla \varphi(x,t) \cdot v(x,t) \right) d\mu_t(x)\, dt = 0$$

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0,T)).$$

Def: (solution of ODE): Given $v : \mathbb{R}^d \times (0,T) \to \mathbb{R}^d$ meas and $x_0 \in \mathbb{R}^d$, $x : (0,T) \to \mathbb{R}^d$ is a solution of

$$\begin{cases} x'(t) = v(x(t), t) \\ x(0) = x_0 \end{cases}$$

if

- $x(t)$ is locally abs cts, $\lim_{t \to 0^+} x(t) = x_0$.
- the ODE holds in the integral sense, i.e.

$$x(t) = x_0 + \int_0^t v(x(s), s)\,ds, \quad \forall t \in [0,T].$$

Fix $v(x,t)$. Suppose that a solution of (ODE) exists $\forall x_0 \in \mathbb{R}^d$. In this case, we may consider the flow map induced by v,

$X_t(y) = X(y,t) = x(t)$, where $x(t)$ is soln w/ $x(0) = y$.

Prop: Fix $v(x,t)$ and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Suppose that solns of (ODE) exist μ_0-a.e. $x_0 \in \mathbb{R}^d$, $\forall t \in [0,T]$.
Let X_t be the corresponding flow map, which is defined μ_0-a.e..

Then define $\mu_t = X_t \# \mu_0$.

Then, if $\int_0^T \int_{\mathbb{R}^d} |v(x,t)|^2 d\mu_t(x)\, dt < +\infty$,

- $\mu_t \in \mathcal{P}_2(\mathbb{R}^d) \quad \forall\, t \in [0,T]$
- (μ, v) is a soln of (CE) on $\mathbb{R}^d \times [0,T]$.

Pf: Last time, we showed second bullet.

Furthermore, $\mu_t \in \mathcal{P}(\mathbb{R}^d) \ \forall\, t \in [0,T]$.

To see that $\int |x|^2 d\mu_t(x) < +\infty \ \forall t \in [0,T]$, recall that, by defn of ODE,

$$|X_t(y)| \leq |y| + \int_0^T |v(X_s(y), s)| ds \quad \mu_0\text{-a.e. } y.$$

$$\int |x|^2 d\mu_t(x) = \int |X_t(y)|^2 d\mu_0(y)$$

$$\leq 2 \int_{\mathbb{R}^d} |y|^2 d\mu_0(y) + 2 C_T \int_0^T \int_{\mathbb{R}^d} |v(X_s(y), s)|^2 d\mu_0(y)\, ds$$

$$= 2\int_{\mathbb{R}^d} |y|^2 d\mu_0(y) + 2C_T \int_0^T \int_{\mathbb{R}^d} |v(x,s)|^2 d\mu_t(x)\, ds \quad \square$$

$< +\infty$

We will now use this Lagrangian perspective to connect Wasserstein geodesics to (CE).

Def: Given a metric space (X, d), $x : [0,1] \to X$ is a (constant speed) geodesic if

$$d(x(t), x(s)) = |t-s|\, d(x(0), x(1))$$

Prop: Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, $\underbrace{\mu_0, \mu_1 \ll \mathcal{L}^d}_{\text{the prop continues to hold without this}}$, there exists a constant speed geodesic between them.

Furthermore, there exists a velocity v s.t. (μ, v) is a soln (CE) and

$$\left(\int |v(x,t)|^2 d\mu_t(x)\right)^{1/2} \equiv W_2(\mu_0, \mu_1) \quad \forall t \in [0,1]$$

The proof of this proposition relies on the following:

Prop: Suppose $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ are abs cts w.r.t. Lebesgue. Let T denote the OT map from μ to ν (for the quadratic cost).
(i) $\gamma = (id \times T)\#\mu$ is the unique OT plan.
(ii) If $\tilde{T}$ is the OT map from ν to μ,

$$T \circ \tilde{T} = id \quad \nu \text{ a.e.}, \quad \tilde{T} \circ T = id \quad \mu\text{-a.e.}$$

In particular, T is left invertible μ-a.e. with $T^{-1} = \tilde{T}$.

Pf: First show (i). We have already shown γ is an OT plan. It remains to show it is unique.

By Brenier's theorem, $\exists\ \varphi \in L^1(\mu)$ proper, convex, lsc s.t. $T = \nabla\varphi$.

Thus by defn, $x^2 = \nabla\varphi(x^1) \in \partial\varphi(x^1)$ γ-a.e.

Fix another OT plan γ'.

By Knott-Smith theorem,

$$x^2 \in \partial\varphi(x^1) \qquad \gamma'\text{-a.e.}$$

Since φ is diff μ-a.e.

$$x^2 = \nabla\varphi(x^1) \qquad \gamma'\text{-a.e.}$$

Thus $\gamma' = (id \times T)\#\mu = \gamma$.

Now, show (ii).

Since $\gamma' = (\tilde{T} \times id)\#\nu$ is an OT plan, by part (i),

$$\gamma' = (\tilde{T} \times id)\#\nu = (id \times T)\#\mu = \gamma$$

Thus,

$$\int |y - T\circ\tilde{T}(y)| \, d\nu(y)$$

$$= \int |x^2 - T(x^1)| \, d\gamma'(x^1, x^2)$$

$$= \int |x^2 - T(x^1)| \, d\gamma(x^1, x^2)$$

$$= \int |T(y) - T(y)| \, d\mu(y) = 0.$$

Arguing symmetrically then gives the result. $\square$

Now, we prove main geodesic thm.

Pf:

Since $\mu_0, \mu_1 \ll \mathcal{L}^d$, $\exists$ OT map T from μ_0 to μ_1.

$id(x) = x$

Define $\mu(t) := \underbrace{((1-t)id + tT)}_{T_t} \# \mu_0$

Note that, if $T = \nabla\varphi$, $T_t = \nabla\left((1-t)\frac{x^2}{2} + t\varphi(x)\right)$.

Thus, by change of variables formula,

$$d\mu_t(y) = \frac{\mu_0}{|\det DT_t|} \circ T_t^{-1}(y)\, d\mathcal{L}^d(y)\Big|_{T_t(\mathbb{R}^d)}$$

Thus $\mu_t \ll \mathcal{L}^d$ for all $t \in [0,1]$.

Since $T_t \# \mu_0 = \mu_t \quad \forall\, t \in [0,1]$,
$(T_t \times T_s) \# \mu_0 \in \Gamma(\mu_t, \mu_s)$.

Thus,

$$W_2^2(\mu_t, \mu_s) \le \int |x^1 - x^2|^2 \, d\big((T_t \times T_s) \# \mu_0\big)(x^1, x^2)$$

$$= \int |T_t(x) - T_s(x)|^2 \, d\mu_0(x)$$

$$= |t-s|^2 \int |x - T(x)|^2 \, d\mu_0(x)$$

$$= |t-s|^2 \, W_2^2(\mu_0, \mu_1)$$

For the other direction of the inequality, note that, $t \ge s$

$$W_2(\mu_0, \mu_1) \le W_2(\mu_0, \mu_s) + W_2(\mu_s, \mu_t) + W_2(\mu_t, \mu_1)$$

$$\le [s + (t-s) + (1-t)] \, W_2(\mu_0, \mu_1)$$

$$= W_2(\mu_0, \mu_1)$$

Thus, we must have "=" throughout

Thus, between any μ_0, μ_1, a constant speed geodesic exists.

To now show that this constant speed geodesic solves (CE), recall from previous prop, T_t is left invertible μ_0-a.e. and T_t^{-1} is OT map from μ_t to μ_0.

Define,

$$v(x,t) = T \circ T_t^{-1}(x) - T_t^{-1}(x)$$

Note that

$$\int |v(x,t)|^2 d\mu_t = \int |v(T_t(y),t)|^2 d\mu_0(y)$$
$$= \int |T(y) - y|^2 d\mu_0(y)$$
$$= W_2^2(\mu_0, \mu_1)$$

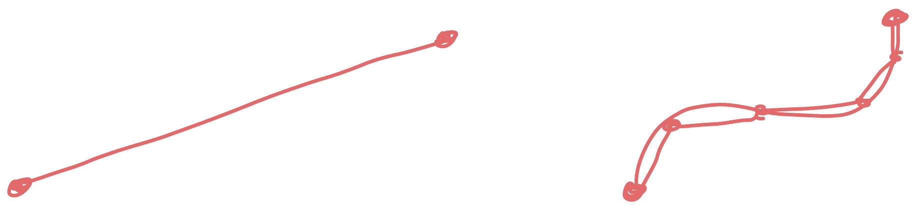
Note that, by defn of T_t and v

$$T_t := ((1-t)id + tT),$$

$$\frac{d}{dt} T_t = T - id$$

$$= v(T_t, t)$$

Thus, $\mu_t := T_t \# \mu_0$, is a weak solution of (CE). $\square$



Now, we will use this PDE characterization of constant speed geodesics to characterize abs cts curves in $(\mathbb{P}_2(\mathbb{R}^d), W_2)$ and prove Benamou-Brenier thm.

We will do this via approximating abs cts curves by (p.w.) geodesics, and we will need a way to conclude the limit is also a soln to (CE).

$m = v\mu$

Def: If $\mu \in \mathcal{P}(\mathbb{R}^d)$, $m \in \mathcal{M}^d_s(\mathbb{R}^d)$, $\mu, m \ll \mathcal{L}^d$, the <u>kinetic energy</u> is

$$\mathcal{B}(\mu, m) = \frac{1}{2}\int \frac{|m(x)|^2}{\mu(x)}\,dx$$ $\sim \int |v|^2 d\mu$

How to extend to general $\mu \in \mathcal{P}(\mathbb{R}^d)$?

Def: For $r \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$f_{\mathcal{B}}(r,x) := \begin{cases} \frac{1}{2}\,|x|^2/r, & \text{if } r > 0 \\ 0 & \text{if } r = x = 0 \\ +\infty & \text{if } r = 0,\ x \neq 0 \text{ or } r < 0. \end{cases}$$

Def (abs cts): Given a metric space (X, d), $x: (0,T) \to X$ is absolutely cts, denoted $x \in AC(0,T;X)$, if $\exists\, g \in L^1([0,T])$ s.t.

$$d(x(t_0), x(t_1)) \leq \int_{t_0}^{t_1} g(s)\,ds \quad \forall\, 0 < t_0 \leq t_1 \leq T.$$

Rmk: if this holds for $g(s) \equiv c \in \mathbb{R}$, then $x(t)$ is Lipschitz continuous.

Def (metric derivative): Given a m.s. (X, d), $x: (0,T) \to X$, the metric derivative is

$$|x'|(t) := \lim_{h \to 0} \frac{d(x(t+h), x(t))}{|h|}$$

generalizes $\left|\frac{d}{dt} x(t)\right|$ for $x(t)$ in a vector space

Fact (Rademacher Thm): Given $x \in AC(0,T;X)$, $|x'|(t)$ exists for a.e. t and

$$d(x(t_0), x(t_1)) \leq \int_{t_0}^{t_1} |x'|(s)\,ds \quad \forall\, 0 < t_0 \leq t_1 \leq T.$$

$W_2(\mu(t_0), \mu(t_1)) \leq \int_{t_0}^{t_1} |\mu'|(s)\,ds$, $|\mu'| \in L^1[0,T]$

Theorem (characterization of abs cts curves in W_2)

(i) Suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ is abs cts.
Then $\exists\, v$ s.t. (μ, v) is a dist soln of cty eqn and $\|v(\cdot,t)\|_{L^2(\mu_t)} \leq |\mu'|(t)$ a.e. $t \in [0,T]$.

$\sqrt{\text{kinetic energy time } t}$

(ii) Conversely, suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ and $\exists\, v$ s.t. $\int_0^1 \|v(\cdot,t)\|_{L^2(\mu(t))}\,dt < +\infty$ and (μ, v) is a dist soln of cty eqn.
Then $\mu(t)$ is abs cts and $|\mu'|(t) \leq \|v(\cdot,t)\|_{L^2(\mu(t))}$ a.e. $t \in [0,T]$.

Pf: "$\Rightarrow$"
Will prove in special case that $\mu(t)$ is concentrated on a cpt set $K \subseteq \mathbb{R}^d$.
WLOG $T = 1$.
Strategy: construct a sequence of simple solns to cty eqn that converge to μ.

Let $\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon^d}\right)$ be a mollifier (cptly supp) with $M_2(\varphi_1) < +\infty$, $\varphi_1 = \varphi$.

Given $k \in \mathbb{N}$, consider discrete time sequence

$$\mu(0/k), \mu(1/k), \ldots, \mu(i/k), \ldots, \mu(k/k)$$

and the mollified sequence

$$\mu^k_{i/k} := \varphi_{1/k} * \mu(i/k), \quad i = 0, \ldots, k$$

B/c this sequence is $\ll \mathcal{L}^d$, the geodesics between each two elements of the sequence are solns to cty eqn. Chaining these geodesics together, we get a dist soln of cty eqn (μ^k, v^k)

reparametrize time so $\mu^k : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$

$$\|v^k(\cdot,t)\|^2_{L^2(\mu^k_t)} = k^2 W_2^2(\mu^k_{i/k}, \mu^k_{i+1/k}) \text{ for } t \in [\tfrac{i}{k}, \tfrac{i+1}{k}]$$

$$\leq k^2 W_2^2(\mu(i/k), \mu(i+1/k))$$

$$\leq \left(k \int_{i/k}^{i+1/k} |\mu'|(s)\,ds\right)^2$$

Jensen's

$$\leq k \int_{i/k}^{i+1/k} |\mu'|^2(s)\,ds$$

Define $m^k(t) := v^k(\cdot,t)\,d\mu^k(t)$, which is a vector measure on $\mathbb{R}^d$ concentrated on a compact set.

Furthermore,

$$|m^k(t)|(\mathbb{R}^d) = \left(\left(\int_{\mathbb{R}^d} |v^k(\cdot,t)|\,d\mu_t^k\right)^2\right)^{1/2}$$

$$\leq \left(\int_{\mathbb{R}^d} |v^k(\cdot,t)|^2\,d\mu_t^k\right)^{1/2}$$

$$\underbrace{\leq k \int_{i/k}^{i+1/k} |\mu'|(s)\,ds}_{\in L^1[0,1]} \qquad \text{for } t \in \left[\tfrac{i}{k}, \tfrac{i+1}{k}\right)$$

By Lemma, up to a subseq, $m^k(t)\,dt \to m(t)\,dt$ narrowly wrt $C_b(\mathbb{R}^d \times [0,1])$

Similarly, μ_t^k converges to μ_t pointwise in time, since.

$W_2(\mu^k_t, \mu_t)$

$$\leq \underset{\text{geodesic}}{W_2(\mu_t^k, \mu_{i/k}^k)} + \underset{\text{mollifier}}{W_2(\mu_{i/k}^k, \mu_{i/k})} + W_2(\mu_{i/k}, \mu_t)$$

$W_2(\mu, \varphi_\varepsilon * \mu) \leq \varepsilon\, M_2(\varphi)^{1/2}$

$$\le \left|t-\tfrac{i}{k}\right| W_2\left(\mu^k_{i/k}, \mu^k_{i+1/k}\right) + \tfrac{1}{k} m_2(\varphi)^{1/2} + \int_{i/k}^{t} |\mu'|(s)\,ds$$

mollifier

$$\le \left|t-\tfrac{i}{k}\right| W_2\left(\mu_{i/k}, \mu_{i+1/k}\right) + \tfrac{1}{k} m_2(\varphi)^{1/2} + \int_{i/k}^{t} |\mu'|(s)\,ds$$

$$\le \tfrac{1}{k} \underbrace{\int_{i/k}^{i+1/k} |\mu'|(s)\,ds}_{\le \| |\mu'| \|_{L^1}} + \tfrac{1}{k} m_2(\varphi)^{1/2} + \underbrace{\int_{t-1/k}^{t} |\mu'|(s)\,ds}$$

$|\mu'| \in L^1 \Rightarrow \forall \varepsilon > 0, \exists \delta$ s.t. $|\Omega| < \delta \Rightarrow \int_\Omega |\mu'| < \varepsilon$.

$\xrightarrow{k \to +\infty} 0$, uniformly int.

In particular, since $\mu^k_t(\mathbb{R}^d) \equiv 1 \in L^1[0,T]$, $\mu^k_t dt \to \mu_t dt$ narrowly wrt $C_b(\mathbb{R}^{d} \times [0,T])$.

Thus, $\forall\, \varphi \in C^\infty_c(\mathbb{R}^d \times [0,T])$

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi(x,t) + v^k(x,t) \cdot \nabla \varphi(x,t)\right) d\mu^k_t\, dt = 0$$

$$\Big\downarrow k \to \infty$$

$$\int_0^T \left(\int_{\mathbb{R}^d} \partial_t \varphi(x,t)\, d\mu_t + \nabla \varphi(x,t)\, dm_t \right) dt = 0.$$

Since $m^k_t = v^k_t\, d\mu^k_t$, $m^k << \mu^k_t$, and

$$B(\mu^k_t, m^k_t) = \int |v^k|^2\, d\mu^k_t$$

$i+1/k$

$$\leq k \int_{i/k} |\mu'|(s)\,ds \quad \text{for } t \in \left[\tfrac{i}{k}, \tfrac{i+1}{k}\right],$$

we have,

$$\int_0^1 |\mu'|(s)\,ds \geq \liminf_{k\to\infty} \int_0^1 \mathbb{B}(\mu_t^k, m_t^k)\,dt$$

Fatou, lsc of $\mathbb{B}$

$$\geq \int_0 \mathbb{B}(\mu_t, m_t)\,dt.$$

Thus $\mathbb{B}(\mu_t, m_t) < +\infty$ a.e., so $\exists\, v_t$ s.t.

$$\mathbb{B}(\mu_t, m_t) = \int |v_t|^2 d\mu_t, \quad m_t = v_t\, d\mu_t. \ \square$$