Lecture 19

Recall:

Def: For $r \in \mathbb{R}$, $x \in \mathbb{R}^d$,

$$f_{\mathcal{B}}(r,x) := \begin{cases} \frac{1}{2}|x|^2/r & \text{if } r>0 \\ 0 & \text{if } r=x=0 \\ +\infty & \text{if } r=0,\ x\neq 0 \text{ or } r<0. \end{cases}$$

Prop: Given $\mu \in \mathcal{P}(\mathbb{R}^d)$, $m \in \mathcal{M}_s^d(\mathbb{R}^d)$, define

$$\mathcal{B}(\mu,m) := \sup_{\substack{f \in C_b(\mathbb{R},\mathbb{R}),\ g \in C_b(\mathbb{R}^d,\mathbb{R}^d) \\ f+\frac{1}{2}|g|^2 \leq 0}} \left\{ \int f\,d\mu + \int g \cdot dm \right\}$$

Then,

(i) $\mathcal{B}(\mu,m)$ is convex, lsc wrt narrow convergence

(ii) if $\mu, m \ll \omega$, ω Borel meas on $\mathbb{R}^d$,

$$\mathcal{B}(\mu,m) = \int f_{\mathcal{B}}(\mu(x), m(x))\,d\omega(x),$$

where $d\mu(x) = \mu(x)\,d\omega(x)$,
$dm(x) = m(x)\,d\omega(x)$

(iii) $\mathcal{B}(\mu, m) = \begin{cases} \frac{1}{2}\int |v|^2 d\mu & \text{if } m \ll \mu \\ & dm = v\, d\mu \\ +\infty & \text{otherwise} \end{cases}$

We now have everything we need to characterize absolutely continuous curves in $(\mathcal{P}_2(\mathbb{R}^d), W_2)$... except the definition of what it means to be an absolutely continuous curve!

Suppose (X, d) is a complete metric space.

Def (abs cts): $x : (a,b) \to X$ is abs cts, denoted $x \in AC(a,b;X)$ if $g \in L^1(a,b)$ s.t.

$$d(x(t_0), x(t_1)) \leq \int_{t_0}^{t_1} g(s)\, ds \quad \forall\ a < t_0 \leq t_1 < b.$$

Remark: If this holds for $g(s) \equiv c$, for $c \in \mathbb{R}$, then x is Lipschitz cts.

Rmk: if x is abs cts, it is cts

Def: (metric derivative): The metric derivative of $x:(a,b)\to X$ is

$$|x'|(t) := \lim_{h\to 0} \frac{d(x(t+h),x(t))}{|h|}$$

generalizes $\left|\frac{d}{dt}x(t)\right|$, for $x(t)$ a curve in a vector space

Prop: For any $x \in AC(a,b;X)$,
(i) $|x'|(t)$ exists for $\mathcal{L}$-a.e. $t\in(a,b)$,
(ii) $g(t)=|x'|(t)$ is admissible in $(*)$
(iii) $|x'|(t) \le g(t)$ $\mathcal{L}$-a.e. for all g satisfying $(*)$.

Pf:

Since $x:(a,b)\to X$ is cts and (a,b) is separable, $x(a,b)$ is separable. Let $\{y_n\}_{n=1}^{+\infty}$ be a dense sequence, and consider

$$d_n(t) = d(y_n, x(t)).$$

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By the reverse triangle inequality, for any choice of g in the defn of AC, we have

$$|d_n(t)-d_n(s)| \overset{(*)}{\le} d(x(t),x(s)) \overset{(**)}{\le} \int_s^t g(r)\,dr.$$

Thus, $t \mapsto d_n(t)$ is abs cts, and

$$d(t) := \sup_{n\in\mathbb{N}} |d_n'(t)|$$

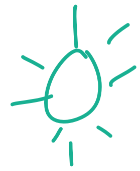
is well-defined $\mathcal{L}$-a.e. Furthermore, if it is defined $t\in(a,b)$,

$$d(t) \overset{(\dagger)}{=} \sup_{n\in\mathbb{N}} \liminf_{s\to t} \frac{|d_n(t)-d_n(s)|}{|t-s|}$$
$$\le \liminf_{s\to t} \frac{d(x(t),x(s))}{|t-s|}$$

Thus by $(**)$ $d(t)\le g(t)$ $\mathcal{L}$-a.e, so $d\in L^1(a,b)$.

To obtain an upper bound on limsup, note that, by density of y_n,

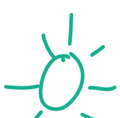
$$\left.\begin{aligned} d(x(t),x(s)) &= \sup_{n\in\mathbb{N}} |d_n(t)-d(s)| \\ &= \sup_{n\in\mathbb{N}} \left| \int_t^s d_n'(r)\,dr \right| \\ &\le \sup_{n\in\mathbb{N}} \int_t^s |d_n'(r)|\,dr \\ &\le \int_t^s d(r)\,dr \end{aligned}\right\}$$



Thus, for $\mathcal{L}$-a.e. t.

$$\limsup_{s\to t} \frac{d(x(s),x(t))}{|s-t|} \le d(t).$$

Therefore $|x'|(t)$ exists $\mathcal{L}$-a.e. t, and $|x'|(t) = d(t)$.

 shows $|x'|(t)$ is admissible choice of g in defn of AC.

 shows that $|x'|(t)$ is minimal. $\square$

Theorem (characterization of abs cts curves in W_2):

(i) Suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ is abs cts. Then $\exists\ v$ s.t. (μ, v) is a weak soln of (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \right)^{1/2} \leq |\mu'|(t) \quad \mathcal{L}\text{-a.e. } t \in [0,T].$$

(ii) Converseley, suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ and $\exists\ v$ s.t.

$$\int_0^1 \left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \right) dt < +\infty$$

and (μ, v) is a soln of (CE).

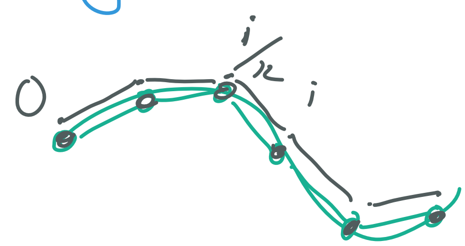
Then, $\mu(t)$ is abs cts and

$$|\mu'|(t) \leq \left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x) \right)^{1/2} \quad \mathcal{L}\text{-a.e. } t \in [0,T]$$

Our proof of (i) relies on this Lemma:

<u>Lemma</u>: Suppose $m^k : [0,T] \to \mathcal{M}_s^d(\mathbb{R}^d)$, $\exists\, K \subset\subset \mathbb{R}^d$ s.t. $m^k_t(K^c) \equiv 0$, and $\exists\, f \in L^1(0,T)$ s.t. $|m^k_t|(\mathbb{R}^d) \leq f(t)$ $\mathcal{L}$-a.e. $t \in [0,T]$.

Then, up to a subsequence, there exists $m : [0,T] \to \mathcal{M}_s^d(\mathbb{R}^d)$ s.t. $dm^k_t(x)\,dt \to dm_t(x)\,dt$ narrowly wrt $C_b(\mathbb{R}^d \times [0,T])$.



Pf:

Today, we will show (i).

For simplicity, we will assume that
$$\exists\, K \subset\subset \mathbb{R}^d \text{ s.t. } \mu_t(K^c) = 0 \ \forall t,$$
and that $T = 1$.
Let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$, for a compactly supported mollifier φ, with $m_2(\varphi) < +\infty$.

For $k \in \mathbb{N}$, consider the "discrete time sequence"

$$\mu(0/k), \mu(1/k), \dots, \mu(i/k), \dots, \mu(k/k).$$

and the mollified sequence

$$\mu^k_{i/k} = \varphi_{1/k} * \mu(i/k)$$

By our previous proposition, since these measures are $\ll \mathcal{L}^d$, there exists a unique geodesic $\mu_i^k(t)$ from $\mu_i^k(0) = \mu^k_{i/k}$ to $\mu_i^k(1) = \mu^k_{i+1/k}$.

Furthermore (μ_i^k, v_i^k) is a soln of (CE) and

$$\left(\int |v_i^k(x,t)|^2 \, d(\mu_i^k)_t(x) \right)^{1/2} = W_2(\mu^k_{i/k}, \mu^k_{i+1/k}) \quad \forall t \in [0,1].$$

Now, we chain these geodesics together, defining

$$\mu^k(t) := \mu_i^k(tk - i) \text{ for } t \in [i/k, i+1/k).$$
$$v^k(t) := v_i^k(tk-i) \cdot k$$

Then (μ^k, v^k) is a weak soln of

$$\partial_t \mu^k + \nabla \cdot (\mu^k v^k) = 0$$

Furthermore, for $t \in [i/k, i+1/k)$

$$\int |v^k(x,t)|^2 d\mu_t^k(x) = k^2 W_2^2(\mu^k_{i/k}, \mu^k_{i+1/k})$$

W2 contracts under mollification

$$\leq k^2 W_2^2(\mu(i/k), \mu(i+1/k))$$

$$\leq \left(k \int_{i/k}^{i+1/k} |\mu'|(s) ds \right)^2$$

$$\leq k \int_{i/k}^{i+1/k} |\mu'|^2(s) ds$$

Define $dm_t(x) := v^k(x,t) d\mu_t^k(x) \in \mathcal{M}_s^d(\mathbb{R}^d)$.

Note that, for $t \in [i/k, i+1/k]$,

$$|m_t^k|(\mathbb{R}^d) = \int |v^k(x,t)| d\mu_t^k(x)$$

$$\leq \left(\int |v^k(x,t)|^2 d\mu_t^k(x) \right)^{1/2}$$

$$\leq k \int_{i/k}^{i+1/k} |\mu'|(s) ds = f(t)$$

Thus, defining

$$f(t) = k \int_{i/k}^{(i+1)/k} |\mu'(s)| ds \quad \text{for } t \in [i/k, \tfrac{i+1}{k})$$

we have,

$$\int_{i/k}^{i+1/k} f(t) dt \leq \int_{i/k}^{i+1/k} |\mu'(s)| ds$$

and

$$\int_0^1 f(t)\, dt \leq \int_0^1 |\mu'|(s) < +\infty,$$

so $f \in L^1(0,1)$.

To be continued...