Lecture 20

Recall:

Suppose $(\mathcal{X}, d)$ is a complete metric space.

Def (abs cts): $x:(a,b) \to \mathcal{X}$ is abs cts, denoted $x \in AC(a,b;\mathcal{X})$ if $g \in L^1(a,b)$ s.t.

$$d(x(t_0), x(t_1)) \leq \int_{t_0}^{t_1} g(s)\,ds \quad \forall\ a < t_0 \leq t_1 < b.$$

Furthermore, if $g \in L^\varphi(a,b)$, then we say $x \in AC^\varphi(a,b;\mathcal{X})$.

Def: (metric derivative): The metric derivative of $x:(a,b) \to \mathcal{X}$ is

$$|x'|(t) := \lim_{h \to 0} \frac{d(x(t+h), x(t))}{|h|}$$

Prop: For any $x \in AC(a,b;\mathcal{X})$,
(i) $|x'|(t)$ exists for $\mathcal{L}$-a.e. $t \in (a,b)$,
(ii) $g(t) = |x'|(t)$ is admissible in $(*)$
(iii) $|x'|(t) \leq g(t)$ $\mathcal{L}$-a.e. for all g satisfying $(*)$

Theorem (characterization of abs cts curves in W_2):

(i) Suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ is $AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$ ~~abs cts~~. Then $\exists\, v$ s.t. (μ, v) is a weak soln of (CE) and

$$\left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x)\right)^{1/2} \leq |\mu'|(t) \quad \mathcal{L}\text{-a.e. } t \in [0,T]$$

(ii) Conversely, suppose $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$ and $\exists\, v$ s.t.

$$\int_0^1 \left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x)\right) dt < +\infty$$

and (μ, v) is a soln of (CE).

Then, $\mu(t)$ is $AC^2(0,T;\mathcal{P}_2(\mathbb{R}^d))$ ~~abs cts~~ and

$$|\mu'|(t) \leq \left(\int_{\mathbb{R}^d} |v(x,t)|^2 \, d\mu_t(x)\right)^{1/2} \quad \mathcal{L}\text{-a.e. } t \in [0,T]$$

Our proof of (i) relies on this Lemma:

<u>Lemma</u>: Given $\{\sigma^k\} \in \mathcal{M}^d_s(\mathcal{X})$ with $\sup_k |\sigma^k|(\mathcal{X}) < +\infty$, $\{\sigma^k\}$ is tight iff it is relatively narrowly cpt.

<u>Pf</u>:

Last time...

For simplicity, suppose $\exists$ $R>0$ s.t. $\mu_t(B_R^c) \equiv 0$ and $T=1$.

For $k \in \mathbb{N}$, consider the "discrete time sequence"

$$\mu(0/k), \mu(1/k), \dots, \mu(i/k), \dots, \mu(k/k).$$

and the mollified sequence

$$\mu^k_{i/k} = \varphi_{1/k} * \mu(i/k).$$

Then, by chaining together the Wasserstein geodesics connecting these points, we constructed

$\mu^k : [0,1] \to \mathcal{P}(\mathbb{R}^d)$
$v^k : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ measurable

where $\mu^k_t(B^c_{R+1}) \equiv 0$, (μ^k, v^k) is a weak solution of (CE) and for $t \in [i/k, i+1/k]$,

$$\int |v^k(x,t)|^2 d\mu_t(x) \overset{☆}{\leq} \left(k \int_{i/k}^{i+1/k} |\mu'|(s) ds \right)^2 \overset{☆}{\leq} k \int_{i/k}^{i+1/k} |\mu'|^2(s) ds$$

Note that, for any $f \in L^1(0,1)$, $f \geq 0$, if

$$f_k(t) = k \int_{i/k}^{i+1/k} f(s) ds \quad \text{for } t \in [i/k, i+1/k],$$

then,

$$\int_{i/k}^{i+1/k} f_k(t) dt = \int_{i/k}^{i+1/k} f(s) ds$$

Thus, for any $(a,b) \subseteq (0,1)$, if $a \in \left[\frac{i_a - 1}{K}, \frac{i_a}{K}\right]$ and $b \in \left[\frac{i_b}{K}, \frac{i_b+1}{K}\right]$,

$$\int_a^b f_K(t)\,dt \le \int_{i_a/K}^{i_b/K} f(s)\,ds + \int_{(a, i_a/K) \cup (i_b/K, b)} f_K(t)\,dt$$

$$\le \int_a^b f(s)\,ds + \int_{(a, i_a/K) \cup (i_b/K, b)} f_K(t)\,dt,$$

where

$$\int_{(a, i_a/K)} f_K(t)\,dt = \left(\frac{i_a}{K} - a\right) K \int_{i_a - 1/K}^{i_a/K} f(s)\,ds \le \int_{i_a - 1/K}^{i_a/K} f(s)\,ds$$

Thus,

$$\limsup_{K \to +\infty} \int_a^b f_K(t)\,dt \le \int_a^b f(s)\,ds.$$

In particular, for all $(a,b) \subseteq (0,1)$,

• taking $f = |\mu'|^2$ and apply ☆

$$\limsup_{k\to+\infty} \int_a^b \int |v^k(x,t)|^2 d\mu_t(x) \leq \int_a^b |\mu'|^2(t)dt$$

• taking $f = |\mu'|$ and apply ☆

$$\limsup_{k\to\infty} \int_a^b \left(\int |v^k(x,t)|^2 d\mu_t(x)\right)^{1/2} dt \leq \int_a^b |\mu'|(t)dt$$

$$\vee\!|$$

$$\limsup_{k\to\infty} \int_a^b \int |v^k(x,t)|\, d\mu_t(x)\; dt$$

Define $\sigma_k \in \mathcal{M}_s^d(\mathbb{R}^d \times [0,1])$ by

$$\sigma_k = v^k d\mu_t^k dt$$

By above,

$$\limsup_{k\to+\infty} |\sigma_k|(\mathbb{R}^d \times [0,1]) \leq \int_0^1 |\mu'|(t)dt < +\infty$$

Since σ_k are compactly supported, by lemma, there exists a subsequence σ_k and $\sigma \in \mathcal{M}_s^d(\mathbb{R}^d \times [0,1])$ s.t.

$$\sigma_k \to \sigma \quad \text{narrowly.}$$

Similarly, we obtain that μ_t^k converges narrowly to μ^k, uniformly in t, since, for $t \in [i/k, i+1/k]$

$$
\begin{aligned}
&W_2(\mu_t^k, \mu_t) \\
&\quad \leq W_2(\mu_t^k, \mu_{i/k}^k) + W_2(\mu_{i/k}^k, \mu_{i/k}) + W_2(\mu_{i/k}, \mu_t) \\
&\quad \leq |t - \tfrac{i}{k}| W_2(\mu_{i/k}^k, \mu_{i+1/k}^k) + \tfrac{1}{k} m_2(\varphi)^{1/2} \\
&\qquad + \int_{i/k}^{t} |\mu'|(s)\, ds \\
&\quad \leq |t - \tfrac{i}{k}| W_2(\mu_{i/k}, \mu_{i+1/k}) + \tfrac{1}{k} m_2(\varphi)^{1/2} \\
&\qquad + \int_{i/k}^{t} |\mu'|(s)\, ds \\
&\quad \leq |t - \tfrac{i}{k}| \int_{i/k}^{i+1/k} |\mu'|(s)\, ds \quad + \quad " \qquad\qquad "\\
&\quad \longrightarrow 0
\end{aligned}
$$

In particular, we also can obtain

$$d\mu_t^k dt \to d\mu_t dt.$$

Thus, $\forall\ \varphi \in C_c^\infty(\mathbb{R}^d \times [0,1])$

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(x,t)\, d\mu_t^k dt + \int_0^1 \int_{\mathbb{R}^d} v^k(x,t) \nabla \varphi(x,t)\, d\mu_t^k(x) dt = 0$$

Sending $k \to +\infty$,

$$\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(x,t)\, d\mu_t dt + \int_0^1 \int_{\mathbb{R}^d} \nabla \varphi(x,t)\, d\sigma(x,t) = 0.$$

We now seek to show $\exists\ v$ s.t.

$$d\sigma = v\, d\mu_t dt.$$

To see this, note that since

$$d\sigma_k = v_k\, d\mu_t^k dt,\ d\sigma_k \ll d\mu_t^k dt.$$

$$\mathcal{B}(\mu_t^k dt, d\sigma_k) = \int_0^1 \int_{\mathbb{R}^d} |v_k|^2 d\mu_t^k dt$$

Using lsc of $\mathcal{B}$,

$$\mathcal{B}(\mu_t dt, d\sigma)$$

$$\leq \liminf_{k\to+\infty} \mathcal{B}(\mu_t^k dt, d\sigma_k)$$

$$\leq \limsup_{k\to+\infty} \int_0^1 \int |v^k(x,t)|^2 d\mu_t(x)\, dt$$

$$\leq \int_0^1 |\mu'|^2(t)\, dt \quad < +\infty$$

Therefore, we may conclude that

$$d\sigma << d\mu_t dt.$$

Hence, $\exists$ $v(x,t)$ meas s.t.

$$d\sigma = v\, d\mu_t dt.$$

Thus, (μ, v) solves (CE).

Finally, for any interval $(a,b) \subseteq (0,1)$, arguing as above

$$d\sigma^k \big|_{\mathbb{R}^d \times [a,b]} \xrightarrow{\text{narrowly}} v \, d\mu_t dt \big|_{\mathbb{R}^d \times [a,b]}$$

$$d\mu_t^k dt \big|_{\mathbb{R}^d \times [a,b]} \xrightarrow{\text{narrowly}} d\mu_t dt \big|_{\mathbb{R}^d \times [a,b]}$$

Again, using lsc of $\mathcal{B}$,

$$\int_a^b \int_{\mathbb{R}^d} |v|^2 d\mu_t dt \leq \int_a^b |\mu'|(t) dt$$