Lecture 6

Recall:

We seek to prove existence of solutions to:
Kantorovich's Optimal Transport Problem
Given $\mu, \nu \in \mathcal{P}(X)$, solve

$$\min_{\gamma : \gamma \in \Pi(\mu,\nu)} \underbrace{\int_{X \times X} d(x^1,x^2)^p \, d\gamma(x^1,x^2)}_{\mathbb{K}_p(\gamma) :=}, \quad p \geq 1$$

Thm (Prokhorov): Given a Polish space (X,d) and $\mathcal{K} \subseteq \mathcal{P}(X)$,
- $\mathcal{K}$ is relatively compact in narrow topology

$\Updownarrow$

- $\mathcal{K}$ is tight. $\forall \varepsilon > 0$, $\exists K_\varepsilon \subset\subset X$ s.t. $\mu(X \setminus K_\varepsilon) \leq \varepsilon$, $\forall \mu \in \mathcal{K}$

Sketch of proof: tight $\Rightarrow$ relatively cpt follows from using tightness to restrict to compact set $X' \subset\subset X$ and using $\mathcal{K}|_{X'} \subseteq B_{E^*}$, for $E = C(X')$, hence relatively weak-$*$ compact by Banach, Alaoglu, Bourbaki

Prop: Given a Polish space (X, d) and $\mu, \nu \in \mathcal{P}(X)$, $\Gamma(\mu,\nu)$ is relatively compact in the narrow topology.

In fact, we can upgrade this to <u>compactness</u> in the narrow topology using the following lemma...

Lemma: Given Polish spaces (X, d_X), (Y, d_Y), and $\{\mu_n\}_{n=1}^{+\infty} \subseteq \mathcal{P}(X)$ narrowly converging to μ, then for any continuous function $t: X \to Y$, $t\#\mu_n$ narrowly converges to $t\#\mu$.

Pf: For any $f \in C_b(Y)$,

$$\lim_{n\to\infty} \int_Y f \, d(t\#\mu_n) = \lim_{n\to\infty} \int_X \underbrace{f \circ t}_{C_b(X)} \, d\mu_n = \int_X f \circ t \, d\mu = \int_Y f \, d(t\#\mu).$$

Prop: Given a Polish space (X, d) and $\mu, \nu \in \mathcal{P}(X)$, $\Gamma(\mu,\nu)$ is compact in the narrow topology.

Pf: It suffices to show that if $\gamma_n \in \Gamma(\mu,\nu)$ converges narrowly to $\gamma \in \mathcal{P}(X \times X)$, then $\gamma \in \Gamma(\mu \times \nu)$. Since π^1, π^2 are cts, the lemma ensures $\pi^i \# \gamma_n \xrightarrow{\text{narrowly}} \pi^i \# \gamma$. Since $\pi^1 \# \gamma_n \equiv \mu$, $\pi^2 \# \gamma_n \equiv \nu$, we have $\pi^1 \# \gamma = \mu$ and $\pi^2 \# \gamma = \nu$, i.e. $\gamma \in \Gamma(\mu,\nu)$.

Next: lsc of $\mathbb{K}_p(\gamma)$

We will, in fact, show lower semicontinuity for a much wider class of integral functionals, i.e. $\gamma \mapsto \int \varphi \, d\gamma$ for φ lsc and bdd below.

To do this, we will first show that any lsc function may be approximated from below by cts functions.

Lemma: Suppose $g : X \to \mathbb{R} \cup \{+\infty\}$ is lsc and bounded below. Then $\exists \{g_k\}_{k=1}^{+\infty} \subseteq C_b(X)$ s.t. $\lim_{k \to +\infty} g_k(x) \nearrow g(x)$ $\forall x \in X$.

Def: A function $g: X \to \mathbb{R} \cup \{+\infty\}$ is <u>proper</u> if $\exists\, x$ s.t. $g(x) < +\infty$.

Def: Given $g: X \to \mathbb{R} \cup \{+\infty\}$, the <u>Moreau-Yosida regularization</u> is given by

warning: multiple common choices of exponent

$$g_k(x) := \inf_{y \in X} g(y) + k\, d(x,y), \quad k \geq 0.$$

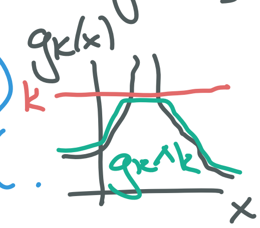
Prop:

(i) If g is proper and bdd below, so is g_k. Furthermore, $g_k \in C(X)$ $\forall\, k \geq 0$.

(ii) If, in addition, g is lsc, then $g_k(x) \nearrow g(x)$ $\forall\, x \in X$.

$g_k(x) \wedge k = \min\{g_k(x), k\}$

(iii) In this case, $g_k \wedge k \in C_b(X)$ and $g_k(x) \wedge k \nearrow g(x)$ $\forall\, x \in X$.



This proves the above lemma.

$\limsup_{n \to \infty} y_n = \overline{\lim}\, y_n$

$\liminf_{n \to \infty} y_n = \underline{\lim}\, y_n$

Pf:

(i) By defn,

$$-\infty < \inf g \leq g_k(x) \leq g(y_0) + k\, d(x, y_0) < +\infty,$$

so g_k is proper and bdd below.

To see $g_k \in C(X)$, suppose $x_n \xrightarrow{d} x$.

On one hand, $\forall\, y \in X$,

$$\overline{\lim}\, g_k(x_n) \le \overline{\lim}\, g(y) + k\,d(x_n, y) = g(y) + k\,d(x, y)$$

Thus, $\boxed{\overline{\lim}\, g_k(x_n) \le g_k(x)}$

On the other hand, we may choose y_n s.t.

$$g(y_n) + k\,d(y_n, x_n) < g_k(x_n) + \frac{1}{n} \quad \forall n \in \mathbb{N}$$

Thus

$$\begin{aligned}
\underline{\lim}\, g_k(x_n) &\ge \underline{\lim}\, g(y_n) + k\,d(y_n, x_n) \\
&\ge \underline{\lim}\, g(y_n) + k\left[d(y_n, x) - d(x_n, x)\right] \\
&\ge g_k(x)
\end{aligned}$$

Thus, $g_k \in C(X)$.

(ii) Now, we show $g_k(x) \nearrow g(x) \ \forall x \in \mathcal{X}$.

Note that $\forall \ k_1 \le k_2$, $g_{k_1}(x) \le g_{k_2}(x) \le g(x)$.

Thus, it suffices to show $\boxed{\underline{\lim}\, g_k(x) \ge g(x).}$

WLOG $\underline{\lim}\, g_k(x) < +\infty$.

Choose y_k s.t. $g(y_k) + k\,d(x, y_k) \le g_k(x) + \frac{1}{k}$.

Then,

$$+\infty > \underline{\lim}\, g_k(x) \ge \underline{\lim}\, \underbrace{g(y_k)}_{\text{bdd below}} + k\underbrace{d(x,y_k)}_{\text{must be going to zero}}$$

Thus $y_k \xrightarrow{d} x$, and by lsc of g,

$$\boxed{\underline{\lim}\, g_k(x)} \ge \underline{\lim}\, g(y_k) + k\,d(x, y_k)$$
$$\ge \boxed{g(x)}$$

(iii) By defn, $g_k \wedge k \in C_b(X)$, since $g_k(x) \nearrow g(x) \ \forall x \in X$, $g_k(x) \wedge k \nearrow g(x) \ \forall x \in X$.

We may now apply the previous proposition to prove...

<u>Thm</u> (Portmanteau): For any $g: X \to \mathbb{R} \cup \{+\infty\}$ lsc and bounded below, the functional $\mu \mapsto \int g \, d\mu$ is lsc wrt narrow conv in $\mathcal{P}(X)$.

that is

$$\mu_n \to \mu \text{ narrowly} \Rightarrow \underline{\lim} \int g \, d\mu_n \ge \int g \, d\mu.$$

<u>Pf</u>: By Moreau-Yosida approximation, $\forall\, k \ge 0$,

$$\underline{\lim} \int g \, d\mu_n \ge \underline{\lim} \int g_k \wedge k \, d\mu_n = \int g_k \wedge k \, d\mu$$

Sending $k \to +\infty$, Fatou's lemma ensures

$$\liminf_{n \to \infty} \int g \, d\mu_n \ge \liminf_{k \to \infty} \int g_k \wedge k \, d\mu \ge \int g \, d\mu. \quad \square$$