I will attempt to have a synchronous Zoom option starting next week.

We seek to prove existence of solutions to: **Kantorovich's Optimal Transport Problem**

Given $\mu, \nu \in P(X)$, solve

$$\min_{\sigma : \sigma \in \Pi(\mu, \nu)} \int_{\times \times \times} d(x^1, x^2) \sigma \, d\gamma(x^1, x^2), \quad p \geq 1$$

**Lemma:** Given Polish spaces $(X, dx)$, $(Y, dy)$, and $\mu_n \equiv \nu$, $n = 1, 2, \ldots \in P(X)$ narrowly converging to $\mu$, then for any continuous function $t : X \to Y$, $t \mu_n$ narrowly converges to $t \# \mu$.

**Prop:** Given a Polish space $(X, \Sigma)$ and $\mu, \nu \in P(X)$, $\Pi(\mu, \nu)$ is compact in the narrow topology.
Lemma: Suppose \( g: X \to [RU^3] \) is lsc and bounded below. Then \( \exists \{g_k\}_{k=1}^{\infty} \leq g(x) \) s.t. \( \lim_{k \to \infty} g_k(x) = g(x) \) \( \forall x \in X \).

Thm (Portmanteau): For any \( g: X \to [RU^3] \) lsc and bounded below, the functional \( \mu \mapsto Sg d\mu \) is lsc wrt narrow conv in \( P(X) \), that is
\[
\mu_n \Rightarrow \mu \text{ narrowly} \Rightarrow \lim \mu_n = \mu
\]

Combining these results, by the Direct Method of the Calculus of Variations, we obtain...

"cost function"

Thm: Given a Polish space \( (X, \mathcal{A}) \), for any function \( c: X \times X \to [RU^3] \) that is lsc and bounded below and \( \mu, \nu \in P(X) \), there exists \( \delta \in P(\mu, \nu) \) satisfying
\[
\delta = \min_{\delta \in P(\mu, \nu)} \int c(x^2, x^2) d\delta(x^2, x^2).
\]
In particular, taking \( c(x^1, x^2) = d(x^1, x^2)^\theta \), we obtain that a solution to Kantorovich's problem exists.

Thus, for any \( \mu, \nu \in P(X) \), there exists an optimal transport plan \( \mathbf{\gamma}_\ast \).

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Why was narrow topology the "right" topology?

Recall:

\( (X, d) \) \( \triangleright \) locally compact metric space

- Banach space \( (C_0(X), \| \cdot \|_\infty) \) \( (Cb(X), \| \cdot \|_\infty) \)
- Dual space \( (\mathcal{M}^s(X), \| \cdot \|_{1tv}) \) \([\text{big space, containing } (\mathcal{M}(X), \| \cdot \|_{1tv})]\)
- Weak-* topology \( \triangleright \) wide topology \( \triangleright \) narrow topology
\( \Gamma(\mu, \nu) \) is narrowly cpt \( \Rightarrow \) widely cpt

But the objective function may not be lsc in wide topology.

Let \( c(x^1, x^2) = d(x^1, x^2) - 1 \).

Consider \( \gamma_n := S(m, n) \in P(\mathbb{R} \times \mathbb{R}) \).

Exercise: \( \gamma_n \to 0 \) in the wide topology

However, \( \lim \left[ Sc(x^1, x^2) d\gamma_n(x^1, x^2) \right] = \lim \left[ S(d(x^1, x^2) - 1) d\gamma_n(x^1, x^2) \right] = -1 < 0 = Sc(x^1, x^2) d\gamma (x^1, x^2) \).

**Moral:** Wide convergence can allow mass to escape to infinity.

However...
Prop: Given a locally compact Polish space $X$ and \( \{\mu_n\}_{n=1}^{\infty} \) and $\mu$ in $\mathcal{P}(X)$ satisfying

\[
\lim_{n \to \infty} S f d\mu_n = S f d\mu \quad \text{for all } f \in C_c(X).
\]

Then, $\mu_n \Rightarrow \mu$ narrowly.

In particular, wide convergence + mass conservation $\iff$ narrow conv.

\[\text{Proof:} \text{ Fix } f \in C_b(X).\]

Since $\{\mu_n\}$ is tight, $\forall k \in \mathbb{N}, \exists K_k \subset X$ s.t. $\mu(X \setminus K_k) = \frac{1}{k}$. By Tietze extension theorem, $\exists \eta_k \in C(X)$ s.t. $\eta_k = 1$ on $K_k$, $\text{supp } \eta_k \subset \subset X$, $0 \leq \eta_k \leq 1$ and $\eta_k \uparrow 1$ pointwise.

Then, $\forall k \in \mathbb{N}$,

\[
\begin{align*}
\liminf_{n \to \infty} S f + \|f\|_{L^\infty} d\mu_n &\geq \liminf_{n \to \infty} S (f + \|f\|_{L^\infty}) \eta_k d\mu_n \\
&= S (f + \|f\|_{L^\infty}) \eta_k d\mu \\
\limsup_{n \to \infty} S f - \|f\|_{L^\infty} d\mu_n &\leq \limsup_{n \to \infty} S (f - \|f\|_{L^\infty}) \eta_k d\mu_n \\
&= S (f - \|f\|_{L^\infty}) \eta_k d\mu.
\end{align*}
\]
Finally, by conservation of mass,

\[ Sf \, d\mu = \|f\|_{L^\infty} + \int f - \|f\|_{L^\infty} \, d\mu \]

\[\begin{align*}
&= \|f\|_{L^\infty} + \lim_{k \to \infty} S(f + \|f\|_{L^\infty}) \, d\mu \\
&\geq \|f\|_{L^\infty} + \limsup_{n \to \infty} \int f - \|f\|_{L^\infty} \, d\mu \\
&= \limsup_{n \to \infty} Sf \, d\mu \\
&\geq \liminf_{n \to \infty} Sf \, d\mu \\
&= \lim_{n \to \infty} Sf \, d\mu \\
&= -\|f\|_{L^\infty} + \lim_{n \to \infty} Sf + \|f\|_{L^\infty} \, d\mu \\
&\geq -\|f\|_{L^\infty} + \lim_{k \to \infty} S(f + \|f\|_{L^\infty}) \, d\mu \\
&= -\|f\|_{L^\infty} + Sf + \|f\|_{L^\infty} \, d\mu \\
&= Sf \, d\mu
\end{align*}\]

Thus, equality must hold throughout. Since \(f\) was arbitrary, \(\mu_n \to \mu\) narrowly. \(\Box\)
Rmk: The distinction between narrow and wide convergence is especially important when $X$ is not locally compact, e.g. $X = C([0,1], \mathbb{R})$ or $X = \mathcal{P}(\mathbb{R}^d)$.

So we solved Kantorovich's problem...

... how does this help us solve Monge's problem?

via the Kantorovich dual problem.

Crash course in convex analysis and optimization

Let $(X, \|\cdot\|_X)$ be a normed vector space.
Let \((X^*, \|\cdot\|_{X^*})\) be its dual space, that is, the set of all bounded linear functionals on \(X\) with

\[
\|y\|_{X^*} = \sup_{x \in X, \|x\| \leq 1} y(x)
\]

Given \(x \in X, y \in X^*\), let \(\langle y, x \rangle = y(x)\)

**Exercise:**
- For any collection of convex functions \((f_i)_{i \in I}\) on \(X\), \(\sup_{i \in I} f_i(x)\) is convex.
- For any collection of lsc functions \((f_i)_{i \in I}\) on \(X\), \(\sup_{i \in I} f_i(x)\) is lsc.

**Def:** Given \(f : X \to RU^{\geq 0} \cup \{\infty\}\) proper, its **conjugate** \(f^* : X^* \to RU^{\geq 0} \cup \{\infty\}\) is

\[
f^*(y) = \sup_{x \in X} \{\langle y, x \rangle - f(x)\}.
\]
Ex: Suppose $x = \mathbb{R}$ and $f(x) = e^x$. Then

$$f^*(y) = \sup_{x \in \mathbb{R}} \{yx - e^x\} = -\inf_{x \in \mathbb{R}} \{e^x - yx\}.$$ 

If $y < 0$, $f^*(y) = +\infty$.
If $y = 0$, $f^*(y) = 0$.
If $y > 0$... then $x \mapsto e^x - yx$ is a convex, differentiable fn, so a critical point is a global minimizer

$$e^{x^*} - y = 0 \iff x^* = \log(y) \iff \left\{ e^{x^*} - yx^* \right\} = -y + y\log y$$

Thus, $f^*(y) = \begin{cases} +\infty & \text{if } y < 0, \\ 0 & \text{if } y = 0, \\ y\log y - y & \text{if } y > 0. \end{cases}$

Exercise: If $f(x) = \frac{1}{p} |x|^p$, find $f^*(y)$. An immediate consequence of the defn is...
Prop (Young's Inequality): \( \forall x \in X, y \in X^* \)
\[
f^*(y) + f(x) \geq \langle y, x \rangle
\]
Another immediate consequence of the defn and the above exercises is...

Lemma: For any \( f: X \to \text{TRUE}^{\geq 0} \) proper, \( f^* \) is convex and lsc.

In a similar way, we may define

Def: Given a norm \( X \) and \( f: X \to \text{TRUE}^{\geq 0} \) proper, its biconjugate \( f^{**}: X \to \text{TRUE}^{\geq 0} \) is

\[
f^{**}(x) = \sup_{y \in X^*} \{ \langle y, x \rangle - f^*(y) \}.
\]

Note that, for all \( f \) proper, \( x \in X \),
\[
\implies f^*(y) + f(x) \geq \langle y, x \rangle \quad \forall y \in X^* \quad \text{(Young)}
\]
\[
\iff f(x) \geq \langle y, x \rangle - f^*(y) \quad \forall y \in X^* \quad \text{(Young)}
\]
\[
\iff f(x) \geq f^{**}(x)
\]
Note that, since $f^{**}$ is always convex and lower semicontinuous, a necessary condition for $f = f^{**}$ is that $f$ is convex and lsc.

In fact, this is sufficient!

**Thm (Fenchel-Moreau):** Given a non-empty $X$ and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ proper,

(i) $f$ convex and lsc $\iff f = f^{**}$

(ii) If $f$ is convex and $f(x_0) < +\infty$, then $f$ is lsc at $x_0$ $\iff f(x) = f^{**}(x_0)$.

_Pf:_ by Hahn-Banach. \qed

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**Exercise:**

- If $x_0$ is a minimizer of $f$, what is $f^*(0)$?
- If $x_0$ is a local minimizer of $f$, that is, there exists a neighborhood $U$ of $x_0$ s.t. $f(x_0) \leq f(y)$ for all $y \in U$, and $f$ is convex, prove that $x_0$ is