EXAMPLES OF ABSOLUTELY CONTINUOUS FUNCTIONS

Def: $(-\infty \leq a < b \leq \infty)$

$F : (a,b) \to \mathbb{R}$ is absolutely continuous, denoted $F \in AC$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for any $\{(a_i, b_i)\}_{i=1}^{n}$ disjoint, $a_i < b_i$,

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon$$

Recall: (part of)

Fundamental Theorem of Calculus: $F : (a,b) \to \mathbb{R}$.

TFAE:

(a) $F \in AC$

(c) F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b])$

and $F(x) - F(a) = \int_a^x F'(t)\,dt$.

Theorem 1:

(Lipschitz)

$\exists M>0$ s.t. $|F(x)-F(y)| \leq M|x-y| \quad \forall x,y \in (a,b)$

$\Longleftrightarrow F \in AC$ and $|F'| \leq M$

Theorem 2:

F is convex $\Longleftrightarrow$ $F \in AC$ on every compact subinterval of (a,b) and F' is increasing (where it is defined.)

Theorem 1: $F : (a,b) \to \mathbb{R}$

$\exists M > 0$ s.t. $\forall x, y \in (a,b)$,

$|F(x) - F(y)| \le M|x-y|$ $\Longleftrightarrow$ $F \in AC$ and $|F'| \le M$

$(\star)$

($\Longrightarrow$) Assume $\exists M$ s.t. $(\star)$ holds.

Let $\varepsilon > 0$. Fix $\delta = \frac{\varepsilon}{M}$. Given $\{(a_i, b_i)\}_{i=1}^{n}$

$$\sum_{i=1}^{n} |b_i - a_i| < \delta = \frac{\varepsilon}{M}.$$ Then,

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| \le \sum_{i=1}^{n} M|b_i - a_i| < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

$\therefore F \in AC$

$|F'| \le M$. Let x be s.t. $F'(x)$ is defined.

$$|F'(x)| = \lim_{y \to x} \frac{|F(y) - F(x)|}{|y-x|}$$

$$\le \lim_{y \to x} \frac{M|y-x|}{|y-x|} = M \quad \therefore$$

$|F'| \le M.$

($\Longleftarrow$) $F \in AC$ and $|F'| \le M$. By FTC,

Assume $x < y$.

$$
\begin{aligned}
|F(y)-F(x)| &= \left|\int_x^y F'(t)\,dt\right| \\
&\leq \int_x^y |F'(t)|\,dt \\
&\leq \int_x^y M\,dt \\
&= M|y-x|. \qquad \square
\end{aligned}
$$

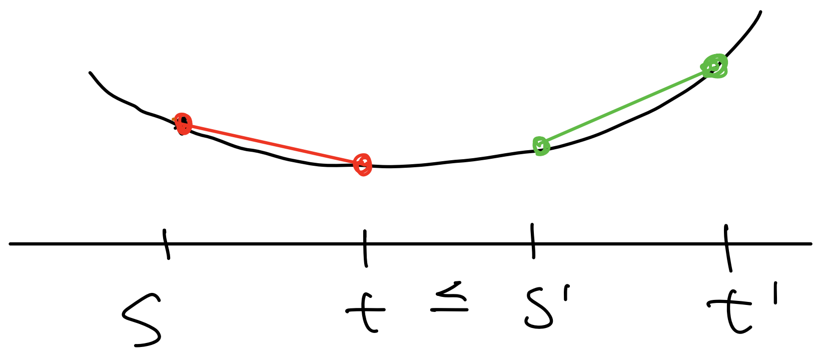
Theorem 2: $F:(a,b)\to\mathbb{R}$.

F is convex $\iff$ $F\in AC$ on every compact subset of (a,b) and F' increasing

Lemma:

F is convex on (a,b) $\iff$ $\forall\ s\le s'<t'$ and $s<t\le t'$,

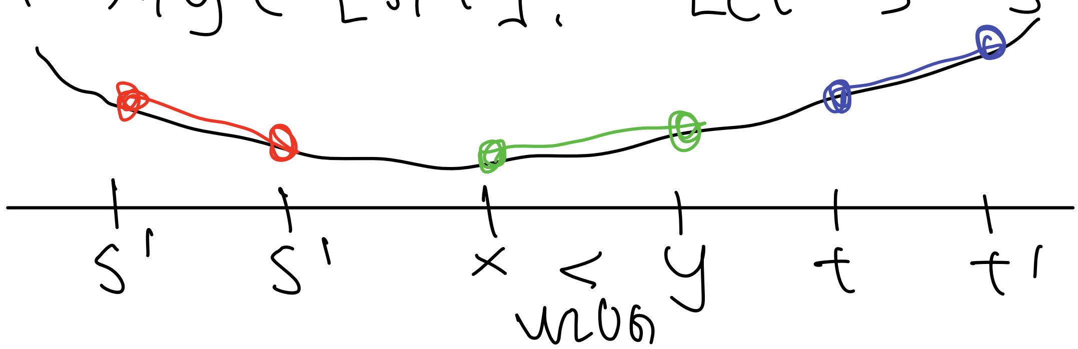
$$\frac{F(t)-F(s)}{t-s}\le\frac{F(t')-F(s')}{t'-s'}$$



slope ≤ green.

($\implies$) By Theorem 1, it suffices to show F is Lipschitz on $[s,t]$.

Let $x,y\in[s,t]$. Let $s'<s<x<y<t<t'$



slope ≤ slope ≤ slope

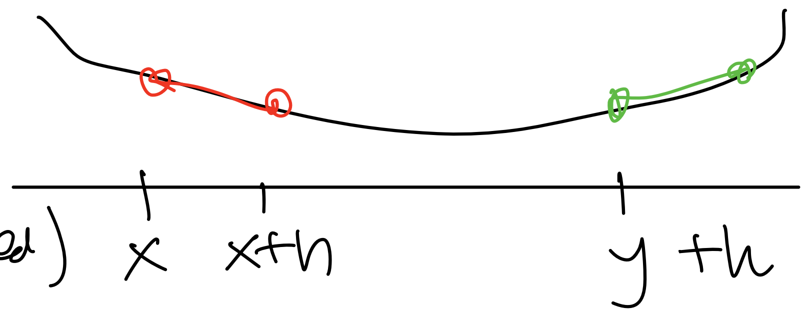
$$\frac{F(s)-F(s')}{s-s'} \le \frac{F(y)-F(x)}{y-x} \le \frac{F(t')-F(t)}{t'-t}.$$

Now let $M = \max\left\{ \left|\frac{F(s)-F(s')}{s-s'}\right|, \left|\frac{F(t')-F(t)}{t'-t}\right| \right.$

$$\Rightarrow \left|\frac{F(y)-F(x)}{y-x}\right| \le M,$$

So $F \in AC$ and $|F'| \le M$ (Thm 1)

$x<y$, (assume F' defined)



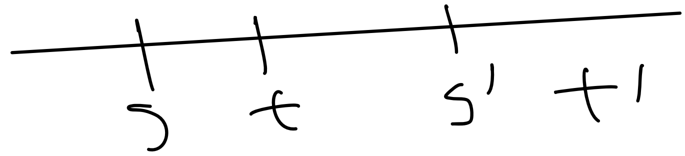
$$\frac{f(x+h)-f(x)}{h} \le \frac{f(y+h)-f(y)}{h}$$

$\downarrow h \to 0$ $\qquad\qquad$ $\downarrow h \to 0$

$$F'(x) \le F'(y).$$

($\Longleftarrow$) Assume $F \in AC$ & F' increasing on all $[s,t] \subseteq (a,b)$.

By FTC:



$$\frac{F(t)-F(s)}{t-s} = \frac{\int_s^t F'(x)\,dx}{t-s} =: L$$

$$\frac{F(t')-F(s')}{t'-s'} = \frac{\int_{s'}^{t'} F'(x)\,dx}{t'-s'} =: R$$

WTS: $L \leq R$.

For a.e. $\ell \in [s,t]$, and a.e. $r \in [s,t']$,

$$F'(\ell) \leq F'(r) \text{ b/c } F' \text{ inc.}$$

So $$\frac{1}{t'-s'} \int_{s'}^{t'} F'(\ell)\,dx \leq \frac{1}{t'-s'} \int_{s'}^{t'} F'(x)\,dx$$

$$F'(\ell) \leq R \quad \text{a.e}$$

Similarly,

$$\frac{1}{t-s}\int_s^t F(x)\,dx \le \frac{1}{t-s}\int_s^t R\,dx$$

$$L \le R. \quad \square$$