Polar Decomposition: $f: \mathbb{R}^n \to \mathbb{R}^m$

$Jf(x) := \sqrt{\det Df(x)^* Df(x)}$ is relatively obscure; not clear how to interact with this information by looking at $Df(x)$.

In DG context, $Jf(x)$ represents volumetric deformation of pulling back inner product in $\mathbb{R}^m$.

Consider the following decomp:

$L: \mathbb{R}^n \to \mathbb{R}^m$ lin.

Then $L = OS$, O is orthogonal / isometry $\mathbb{R}^n \to \mathbb{R}^m$; $O^*O = I$.

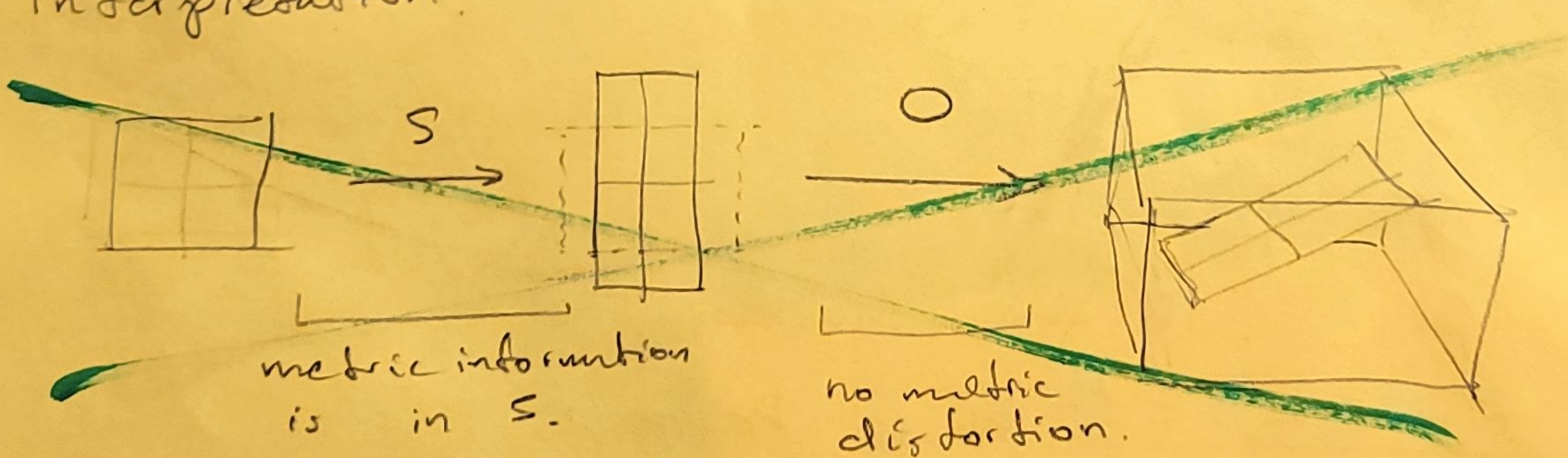
S is symm., if L inv. the S is inv. and pos. defn. (in general, PSD)

Pf (in invertible case):

$S = (L^*L)^{1/2}$.

$O := LS^{-1}$, $O^*O = S^{-1}S^2S^{-1} = I$.

Interpretation:



in particular: $|Df(x)v| = |OSv| = |Sv|$

and $\det[Df(x)^* Df(x)] = \det[SO^*OS] = \det[S]^2$

So $Jf(x) = |\det S|$.

LLAL:

Lemma: Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ lsz. $t > 1$

Denote $A_{reg} := \{ Df(x) \text{ exists}, \ Jf(x) > 0 \}$

$\uparrow$ S is invertible.

Then $\exists \{E_k\}_{k=1}^{\infty}$ Borel, $T_k \in \mathbb{R}^{n \times n}$ symm., inv. s.t.

i. $f|_{E_k}$ is injective (so, invertible).

ii. $\text{Lip}(f|_{E_k} \circ T_k^{-1}) \leq t$, $\text{Lip}(T_k \circ f|_{E_k}^{-1}) \leq t$

$(\sim \text{Lip}(f|_{E_k} \circ T_k^{-1}) \geq t^{-1})$

iii. $t^{-n} |\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|$.

Rmk: ii. is the volumetric deformation control from the linear approximation of T_k to f on E_k.

iii. Allows us to compare the linear approx. to $Jf|_{E_k}$. (In C^1 case this is unnecessary).

Pf: Consider C dns., count. in A_{reg}.

Ω dns., count. in $\mathbb{R}^{n \times n}$ symm. inv.

Fix $t^{-1} + \epsilon < 1 < t - \epsilon$. Define $E_k \sim E(c, T, i)$:

$E(c, T, i) :=$ { $b \in A_{reg}$ |

- $b \in B(c, 1/i)$ ← locality.
- $(t^{-1} + \epsilon)|Tv| \leq |Df(b)v| \leq (t - \epsilon)|Tv|$
  $\sim (t^{-1}+\epsilon)|v| \leq |S \circ T^{-1} v| \leq (t-\epsilon)|v|$.
- $\forall a \in B(c, 2/i)$,
  $|f(a) - f(b) - Df(b)(a-b)| \leq \epsilon |T(a-b)|$
  $\sim o(a-b) \leq \frac{\epsilon}{\|T^{-1}\|} |a-b| \leq \epsilon |T(a-b)|$.

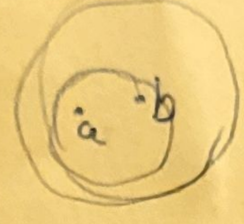
}

For $c \in C$, $T \in \Omega$, $i \in \mathbb{N}$.

i.e. bound lin. error + approximation error.

It is clear that every pt. $b \in A_{reg}$ has membership in some such set $E(c, T, i)$, so covering is shown.

If $a, b \in E(c, T, i)$, $a \in B(c, 1/i) \subset B(b, 2/i)$, so:



$$|f(a) - f(b)| \le |f(a) - f(b) - Df(b)(a-b)| + |Df(b)(a-b)|$$
$$\le ((t-\epsilon) + \epsilon)|T(a-b)| = t|T(a-b)|.$$

Sim. $|f(a) - f(b)| \ge t^{-1}|T(a-b)|$, implying injectivity of f. (i).

Also, $|f \circ T^{-1}(a) - f \circ T^{-1}(b)| \le t|a-b|$, this is condition ii.

Finally, we have $|Df(b)v| = |Sv|$,

so $S(B_1(0)) \subset (t-\epsilon)T(B_1(0))$

$\rightarrow \mathcal{L}^n(S(B_1(0))) \le \mathcal{L}^n((t-\epsilon)T(B_1(0)))$

$\rightarrow |\det S| \le t^n |\det T|$. This is iii. ∎

These notes are based on Evans-Gariepy.