

## Campanato's Criterion Presentation

### Ω Campanato's criterion

Let  $n \geq 1$ ,  $p \in [1, \infty)$ , and  $\gamma \in (0, 1]$ . Then there exists a constant  $C(n, p, \gamma)$  with the following property: if  $u \in L^p(B)$ ,

$$(u)_{x,r} = \frac{1}{|B \cap B(x,r)|} \int_{B \cap B(x,r)} u \quad x \in B, r > 0$$

and there exists a constant  $\kappa$  for which

$$\left( r^{-n} \int_{B \cap B(x,r)} |u - (u)_{x,r}|^p \right)^{1/p} \leq \kappa r^\gamma \quad (1)$$

holds for all  $x \in B$ , then there exists a function  $\bar{u} : B \rightarrow \mathbb{R}$  such that  $\bar{u} = u$  almost everywhere on  $B$  and

$$|\bar{u}(x) - \bar{u}(y)| \leq \kappa C(n, p, \gamma) |x - y|^\gamma \quad (2)$$

for all  $x, y \in B$ .

- Taking  $B$  to be the unit ball, but we can rescale whenever  $u$  is compactly supported.

### Notation

We denote by  $\omega_n$  the Lebesgue measure of a unit ball in  $\mathbb{R}^n$ .

### ⓘ Remark 1

For  $x \in B$  there exists  $c(n) > 0$  for which

$$c(n)r^n \leq |B \cap B(x,r)| \leq \omega_n r^n.$$

A (possibly generous) lower bound on  $c(n)$  is  $2^{-n}\omega_n$ , as  $B(\frac{x}{2}, \frac{r}{2}) \subset B \cap B(x,r) \subset B(x,r)$ .

### Ω Lebesgue points theorem

If  $\mu$  is a Radon measure on  $\mathbb{R}^n$ ,  $p \in [1, \infty)$  and  $u \in L^p_{\text{loc}}(\mathbb{R}^n, \mu)$ , then for  $\mu$ -a.e.  $x \in \mathbb{R}^n$

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u(x) - u|^p d\mu = 0.$$

By the Lebesgue points theorem and the above remark, we find that for  $u \in L^p(B)$  we have

$$\lim_{r \rightarrow 0^+} r^{-n} \int_{B \cap B(x, r)} |u - (u)_{x, r}|^p = 0$$

for a.e.  $x \in B$ , i.e. this holds for every Lebesgue point of  $1_B u \in L^1(\mathbb{R}^n)$ .

### $\Pi$ Jensen's inequality

Let  $\phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be convex, where  $I$  is an interval and we assign values  $\phi(x)$  to  $x \in \partial I - I$  as the limit of  $\phi$  at  $x$ . Let  $f$  be a measurable function from a probability space into  $I$ . If  $f$  and  $\phi(f)$  are integrable, then

$$\phi \left( \int_{\Omega} f d\mu \right) \leq \int_{\Omega} \phi \circ f d\mu.$$

As an aside, for  $f$  non-negative and Lebesgue integrable on a closed interval  $I$  (possibly with  $\mathcal{L}(I) \neq 1$ ) we can adjust the result via

$$\phi \left( \frac{1}{\mathcal{L}(I)} \int_I f \right) \leq \frac{1}{\mathcal{L}(I)} \int \phi \circ f.$$

Similar results hold for  $L^p$  functions, mutatis mutandis. Another (perhaps more familiar) case of Jensen's inequality states the following for  $p \geq 1$ :

$$\left| \sum_{i=1}^n x_i \right|^p = n^p \left| \sum_{i=1}^n \frac{1}{n} x_i \right|^p \leq n^p \sum_{i=1}^n \frac{1}{n} |x_i|^p = n^{p-1} \sum_{i=1}^n |x_i|^p$$

### $\textcircled{i}$ Remark 2

If  $u \in L^p(B)$ ,  $\bar{u} : B \rightarrow \mathbb{R}$  satisfies (2) for some constant  $\kappa'$ , and  $\bar{u} = u$  a.e. on  $B$ , by Jensen's inequality and the fact that  $(\bar{u})_{x, r} = (u)_{x, r}$  we obtain

$$|\bar{u}(x) - (u)_{x,r}| \leq \left( \frac{1}{|B \cap B(x,r)|} \int_{B \cap B(x,r)} |\bar{u}(x) - \bar{u}(y)|^p dy \right)^{1/p} \leq \kappa' r^\gamma$$

for all  $x \in B$  (which implies (1), setting  $\kappa = \kappa'$ ).

- This gives the converse! That is, it says that Campanato's criterion having it's conditions satisfied is an equivalent statement to the conclusion.

### ⊗ Proof of Campanato's Criterion

Let  $c(n)$  be the positive constant as in Remark 1. Setting  $v_r(x) \stackrel{\text{def}}{=} (u)_{x,r}$ , then for every  $r < R$  and  $x \in B$  it holds that

$$c(n)r^n |v_r(x) - v_R(x)|^p \leq 2^{p-1} \left( \int_{B \cap B(x,r)} |u - (u)_{x,r}| + \int_{B \cap B(x,R)} |u - (u)_{x,R}|^p \right).$$

Then by the decay condition (1) we have

$$|v_r(x) - v_R(x_0)| \leq C(n,p)\kappa \left( \frac{R}{r} \right)^{n/p} R^\gamma. \quad (3)$$

Setting  $r_k = 2^{-k}r$  and repeatedly applying (3), we find that

$$|v_{r_k}(x) - v_{r_h}(x)| \leq \sum_{j=h}^{k-1} |v_{r_{j+1}}(x) - v_{r_j}(x)| \leq C(n,p)\kappa \sum_{j=h}^{k-1} (2^{-\gamma})^j r^\gamma \quad (4)$$

for  $0 \leq h < k$  when  $x \in B$ . In particular, when  $h = 0$  and  $k \rightarrow \infty$ , Remark 1 applied to (4) yields

$$|u(x) - v_r(x)| \leq C(n,p,\gamma)\kappa r^\gamma \quad (5)$$

whenever  $x$  is a Lebesgue point of  $u$ . Now let  $x, y \in B$  and set  $r = |x - y|$ . Let  $\eta(n) > 0$  satisfy  $|B(x,r) \cap B(y,r)| = \eta(n)r^n$  and let  $\eta'(n) < \eta(n)$  satisfy  $\eta'(n)r^n \leq |B(x,r) \cap B(y,r) \cap B|$ . Then

$$\eta'(n)r^n |v_r(x) - v_r(y)|^p \leq 2^{p-1} \left( \int_{B \cap B(x,r)} |u - (u)_{x,r}|^p + \int_{B \cap B(y,r)} |u - (u)_{y,r}|^p \right),$$

which alongside (1) implies  $|v_r(x) - v_r(y)| \leq C(n,p)\kappa |x - y|^\gamma$ . By the triangle inequality and (5), for Lebesgue points  $x, y \in B$  of  $u$  we have

$$|u(x) - u(y)| \leq C(n,p,\gamma)\kappa |x - y|^\gamma. \quad (6)$$

Furthermore, (4) implies that for every  $r > 0$  the sequence of continuous functions  $(v_{r_k})_{k \in \mathbb{N}}$  is Cauchy with respect to uniform convergence on  $B$ , so by completeness there exists a continuous function  $\bar{u} : B \rightarrow \mathbb{R}$  which is the pointwise limit of  $(v_{r_k})$ . By the Lebesgue points

theorem,  $\bar{u} = u$  at every Lebesgue point of  $u$  (hence  $\bar{u}$  is independent of our original choice of  $r$ ). Then by (6),  $\bar{u}$  will satisfy (2) for all Lebesgue points  $x, y$  of  $u$ . By the continuity of  $\bar{u}$ , (2) holds for every  $x, y \in B$ .